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Essays In Adaptive Learning And Mean-Square Stability In Regime Switching Models

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ESSAYS IN ADAPTIVE LEARNING AND MEAN-SQUARE STABILITY IN REGIME SWITCHING MODELS

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DEDICATION

To my parent: thank you for nurturing the sparks of curiosity which grew to into the flames of knowledge

To Meghan, my firefly: "You can never cross the ocean unless you have the courage to lose sight of the shore"

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1 "MEAN-SQUARE STABILITY AND ADAPTIVE LEARNING IN REGIME SWITCHING MODELS"

INTRODUCTION

Recently, the Federal Reserve has experienced some changes in the top leadership. Most notably, we witnessed the confirmation of a new fed chief, Janet Yellen. Moreover, Federal Reserve Governor Jeremy Stein announced his resignation from the Board of Governors while Cleveland Fed President Sandra Pianalto announced her position, both effective in May 2014. These resignations place a watchful eye on the next nominations whom will fill positions on the Federal Open Market Committee (FOMC). Stein and Pianalto were characterized by their neutrality on the committee, neither being too "hawkish" or "dovish" when deciding monetary policy. Some believe that the FOMC could be headed for a shift in the balance of power, which would result in policy formation different from what we have been witnessing.

These shifts or swings can be interpreted as regime changes in monetary policy. Although the idea of changing regimes isn't novel, it is still an important dynamic when trying to understand and create monetary policy. To model the stochastic nature of regimes, economist have been using variables characterized by Markovian properties in order to help understand the dynamics of the model under seemingly random transitions. In recent years, economists have focused on the stochastic nature of interest rate movements, using Taylor's (1993) general result, the Taylor Principle. Guided by the stability conditions set forth by Lucas, the Taylor Principle describes the set of responses from the monetary authority for long-run stability. Yet recent studies by Farmer, Waggoner, and Zha (2009) and Cho (2012, 2014) find that the Taylor principle is complicated by the use of Markov switching regimes and that long-run stability under these conditions can be tumultuous. In response to these findings, this paper focuses on the conditions necessary to ensure stability by introducing a new methodology for characterizing long-run stability, Mean-Square Stability (MSS).

Using the model of monetary hyperinflation proposed by Cagan (1956), I am able to build on the ideas presented by Branch, Davig, and McGough in their recent paper which explores the relationship between Markov switching models and adaptive learning. Since the Federal Reserve appears to be headed for a change in regimes, this is an appropriate time in macroeconomics to analyze how regime changes affect the expectations of economic agents. Modern macroeconomics relies heavily on rational expectations equilibria that it still remains the benchmark for alternative measures of expectations. Adaptive learning is an important deviation from the norm, since it allows agents to posit the value of future, key parameters, unlikely known even by a trained economists. The inclusion of adaptive learning into economic modeling has offered an appropriate alternative to rational expectations since its comprehensive introduction by Evans and Honkopohja in 2001. While more recent literature has begun to integrate adaptive learning within the context of Markov switching regimes. Moreover, this paper questions the robustness of E-Stability or stability under adaptive learning, by exploring the relationship that mean-squared stability has to traditional forms of stability under rational expectations.

While using the method of mean-square stability, the primary result of this paper is that Markov switching model equilibria are learnable in the sense that they are E-stable. This is an important result as it provides tractable procedures and outcomes for determinacy. Furthermore, using a simple open economy model, I outline the determinate and indeterminate set of parameter values guiding monetary policy.

The remainder of the paper is organized as follows. The next section dives into the recent and pertinent literature surrounding these topics while Section 2 describes the Cagan model and explains the role of both adapted learning and the Markov-switching parameters. In section 3, I explore the role of Mean-Square Stability in evaluating determinacy and begin to address the connection to the Minimal State Variable solution. Section 4 provides a detailed analysis of adaptive learning and provides the conditions necessary for long-run stability. Conclusions and further discussion are in Section 5. The appendices house the details of the model, methods and results that have been employed.

LITERATURE REVIEW

The literature surrounding regime switching models is robust. Recent monetary policy papers, spanning the time of Volker to Bernanke, have focused primarily on the empirical analysis of the Taylor principle. Davig and Leeper (2007) in their AER paper analyze a conventional new Keynesian model supported by a long-run Taylor principle using an evolving monetary policy. Using a one-county new Keynesian model, they allow monetary policy parameters to fluctuate according to a Markov process. What Davig and Leeper create is a baseline for both empirical and theoretical work which looks at the combination of policy parameters required to find a unique determinate equilibrium. Moreover, they set up a simple closed economy to examine the empirical practices found in changing monetary policy. They find that a determinate equilibrium depends greatly on the combined magnitude of monetary policy parameters. Davig and Leeper conclude that even while using parameter values consistent with providing determinacy to their model, indeterminacy can exists due in part to persistent regimes but also because there can be expectation changes regarding future regimes. Ultimately, they find that these issues can contribute to dramatic increases in inflation volatility, as well as, important parameter choices endowing their model with a unique equilibrium. These sentiments are also echoed by a couple of recent papers from Farmer, Waggoner, and Zha [FWZ] (2010) and Branch, Davig, and McGough (2013).

FWZ, extend the analysis to permit the inclusion of forward looking model components. This extension is important since many rational expectations frameworks include forward looking macroeconomic variables. Here they build a guideline for finding the forward solution for a one-county new Keynesian economy. By including the rational expectations framework, FWZ describe the solution to their model as the minimal state variable solution [MSV] as proposed by McCallum (1983). The MSV solution provides a stable, bounded, unique solution which coincides with the theory governing the Taylor rule. That is, the response of the monetary authority when setting interest rates must be larger than one-for-one to the change in inflation. One important conclusion to their analysis is that FWZ produce a parameterization that yields a seemingly determinant equilibrium. Yet upon closer inspection their solution exhibits an indeterminacy that can exist even when monetary policy follows the Taylor rule. This is because of the interaction between the Markov switching regimes. This spillover between regimes is not captured with traditional determinacy analysis and requires a novel approach to the problem. The exposure of this flaw, coupled with the importance of expectations are the main motivators for this paper. This paper bridges the gap in the literature, with a new understanding of determinacy among Markov switching models as a result from the exploitation of the engineering literature.

The most comprehensive paper to date, which comprises both monetary policy and adaptive learning, within a Markov switching regime, has been recently published by Branch, Davig, and McGough (2013). Branch et al. build on the conclusions of Davig and Leeper and understand that agent expectations play an important role in the long-run equilibrium in a similar new Keynesian framework. As a result they incorporate the use of adaptive learning to model misspecification in an agent's understanding of the deep parameters which govern key endogenous outcomes. Relying on the foundations of a MSV solution with a "bubble" component, Branch et al. outline a solution method and determinacy conditions for two types of forward looking expectations models: History Dependent Regimes [HDR] and Regime Dependent Equilibrium [RDE]. Somewhat intuitive, the HDR is guided by the notion that an agent conditions their expectations on past regimes whereas the RDE sees agents condition only on current ones. These forward looking models, acknowledge the possibility of but ultimately exclude bubbles, i.e. self-fulfilling expectations, as an attainable equilibria. Their paper highlights a unique, stable MSV solution which coincides with a stable equilibrium under adaptive learning. As an empirical test, they develop a simple one-country, new Keynesian example to see if regime switching equilibria are learnable. Their parameterization fails to find an indeterminacy but does concede one can exist. What they forgo becomes an instrumental conclusion of this paper: the necessary conditions for finding the set of indeterminate and determinate solutions. Explored extensively in the engineering literature, mean-square stability is a much stronger set of conditions than commonly used conditions outlined by Blanchard and Khan (1980). MSS takes into account the complexity of the Markov switching regimes and extends the conditions to capture the spillover effects each regime imposes, something Farmer et al. previously recognized. By extending the model from a one-country closed economy to one that is open, I am able to include a parameter sensitive to changing exchange rates within the monetary policy function. Further testing the validity of the Taylor rule.

Taylor himself, in a 2001 AER paper, argues that exchange rates are indeed imbedded into the policy equation of a closed economy, similar to the way new Keynesian models, explored by Branch et al. and Farmer et al., obviously fail to capture any importance they may play. Moreover, he argues that modern monetary policy often includes exchange rate parameters but systematically turn the sensitivity to zero, if they follow the so called Taylor rule that he proposed in 1993. Here Taylor responds to varying criticism over exchange rate inclusion in monetary policy. The most vocal was Obstfeld and Rogoff (1995) who argued that "rule of thumb" changes may be a better monetary policy rule. That is the positive fluctuations in the exchange rate require a negative response from interest rates. Furthermore, they argue that since the theoretical purchasing power parity fails to hold up in the short run as well as over a longer time frame, reactions to exchange rates may be undesirable. But Taylor smartly dissects the discouraging findings and argues that at the very least there are indirect consequences from changing exchange rates. What could be a motivating factor for including the changes in the exchange rate is the stochastic nature of forward looking domestic variables, inflation and output. The model presented in my paper accepts the empirical limitations of PPP, but continues to find the role exchange rates play in monetary policy.

Two recent attempts at compiling a model which includes both learning and Markov-switching parameters comes from Bask (2006) and Ellison, Sarno, and Vilmunen (2007). Ellison et al. provide a theoretical interpretation of the two-country model presented in similar fashion to Walsh (2003). They recognize that by using a more complex global economy, the model is able to capture perceived informational spillovers created by both central banks. Coordination between banks appears to be the most valuable resource for creating a stable economy but it also allows each central bank to exploit the exchange rate for their own advantage. In order to capture this, they build into their model a Baysian learning mechanism that permits central banks to learn how to best exploit the exchange rate. They believe: "any benefit from one central bank learning how to exploit the other then needs to be weighed against the cost of the other central bank also learning how to exploit." One criticism of their model that I address in this paper is that they greatly simplifies the role exchange rates play. They introduce exchange rates into both the aggregate demand and supply equation as endogenous drivers of output but then they remove exchange rates from these equations through the Uncovered Interest Parity (UIP) condition. The UIP links the differential in real interest rates to the difference in the exchange rate today from expected future exchange rates. Moreover, exchange rates do not appear in the Fisher equation, their monetary policy rule, thus rendering the sensitivity of each monetary authority to the fluctuations of exchange rates nonexistent. Furthermore, since their monetary policy rule is assumed to not produce any systematic biases, they conclude that the expectation of future variables, like home and foreign inflation levels, will be zero. It follows then, that the future expected exchange rate is also zero. Completely removing any trace of the exchange rate. Through these decisions the exchange rates acts exogenously and is not needed in their analysis of fluctuating policy and regime changes.

Bask presents a simple open economy, similar to the model developed by Gali and Monacelli where the exchange rate is imbedded into the new Keynesian Phillips curve and the aggregate demand curve. Moreover, they support the model with an uncovered interest rate parity condition similar to what I present in this paper. Lastly, Bask investigates a variety of Taylor policy equations that includes the exchange rate differential in a lagged, contemporaneous, and forward looking positions. The principle conclusion presented by Bask is that monetary policy sensitive matters. Specifically, the monetary authority can ignore exchange rate fluctuations as long as the reaction to inflation is sufficiently large. Although this treatment from Bask includes E-stability it fails to include the possibility of Markov switching regimes.

One of the main conclusions of this paper is finding the conditions necessary to ensure mean-square stability under adaptive learning, thus ensuring E-Stability. The foundation of MSS in economics comes from a recent paper by Farmer, Waggoner and Zha (2009). In their paper, they begin to analyze the role regime switching models have had in recent years and build a framework for the necessary and sufficient conditions to "determine if the parameters of a Markov-switching rational expectations model lead to a determinate equilibrium." Using a forward looking, reduced form model similar to what I use in this paper, I am able to use the necessary and sufficient conditions and apply them to a different class of expectations. One of the main conclusions of my work is the formation of a theorem which reveals that MSV solutions which are mean-square stable are also E-stable under adaptive learning.

THE CAGAN MODEL OF HYPERINFLATION

For the purpose of this analysis, I employ the standard Cagan model of monetary hyperinflation. The framework is used primarily because it is easily tractable and features a stochastic, forward-looking component. The model is outlined as follows,

$$\frac{M_t}{P_t} = \bar{Y}e^{-\alpha(\bar{r}+E^*_{t,t+1}\pi_{t+1})} \qquad (1)$$

where *M* and *P* denote the level of money and process. The expectations operator in Cagan's model is assumed to be adaptive expectations. That is the agent makes expectations of time *t*+1, during time *t* using a weighted average of current and past price levels. For my analysis, the expectations operator coincides with adaptive learning.

$$m_t - p_t = (\log \bar{Y} - \alpha\bar{r}) - \alpha\, E_t^{*}\pi_{t+1}$$

$$m_t - p_t = (\log \bar{Y} - \alpha\bar{r}) - \alpha\,(E_t p_{t+1} - p_t)$$

The level of output, $\bar{Y}$, and the real interest rate, $\bar{r}$, are assumed to be constant. For simplicity, the constant term can be removed by setting $\bar{Y} = \bar{r} = 0$.

$$m_t - p_t = -\alpha(E_t p_{t+1} - p_t) \quad (2)$$

where *m* and *p* are the log of money and the price level. The left-hand side of equation (2) represents the log of real money.

$$p_t = \frac{1}{1+\alpha} m_t + \frac{\alpha}{1+\alpha} E_t p_{t+1} \quad (3)$$

where $\alpha > 0$ and the intercept term can be dropped. Cagan identifies α as coefficient on the velocity of money. In this regard, one would expect the velocity to be positive as interest rates increase. This is because money should turn over more quickly as the opportunity cost of holding money rises.

The main result of this model is that the forward solution exists as long as the limit of expected future prices goes to zero and that the limit of money supply is finite. Therefore when the no bubble conditions are met, prices depend on the velocity of money and the money supply. I amend Cagan's assumption that the velocity of money is constant to an assumption that the velocity of money is state dependent. Where s_t represents a possible *m* state Markov process taking values {1,…..,*m*},

$$p_t = \frac{1}{1+\alpha(s_t)} m_t + \frac{\alpha(s_t)}{1+\alpha(s_t)} E_t p_{t+1} \quad (4).$$

For this analysis, I assume that s_t is only a two state Markov process and that s_t evolves according to the transition matrix,

$$P = \begin{bmatrix} p_{11} & 1 - p_{11} \\ 1 - p_{22} & p_{22} \end{bmatrix},$$

where $P = (p_{ij}) for\ i,j = 1,2$ with p_{ij} being the probability that $s_t = j$ given that $s_{t-1} = i$. The characteristics of the transition matrix are taken to be recurrent and aperiodic implying a unique stationary distribution.

The model simplifies to the reduced form, non-linear, expectational difference equation with form[1]

$$y_t = \beta(s_t)E_t y_{t+1} + \gamma(s_t)r_t,$$

where y_t is the $n\,x\,1$ vector of endogenous variables, $\beta(s_t)$ and $\gamma(s_t)$ are assumed to be invertible, conforming matrices, dependent on the Markov process s_t.

Further simplifying the Cagan model requires a linearization of the equations. Similar to the framework developed by Branch, Davig, and McGough (2013) conditioning the structural form of the model on each regime, s_t, creates the following system,

$$y_{1t} = \beta_1 p_{11} E_t y_{1\,t+1} + \beta_1 p_{12} E_t y_{2\,t+1} + \cdots + \beta_1 p_{1m} E_t y_{m\,t+1} + \gamma_1 r_t,$$

$$y_{2t} = \beta_2 p_{21} E_t y_{1\,t+1} + \beta_2 p_{22} E_t y_{2\,t+1} + \cdots + \beta_2 p_{2m} E_t y_{m\,t+1} + \gamma_2 r_t,$$

$$\vdots$$

$$y_{mt} = \beta_m p_{m1} E_t y_{1\,t+1} + \beta_m p_{m2} E_t y_{2\,t+1} + \cdots + \beta_m p_{mm} E_t y_{m\,t+1} + \gamma_m r_t,$$

this system, now linear can be re-written in a reduced form as

$$\hat{y}_t = M E_t \hat{y}_{t+1} + \eta r_t. \qquad (5)$$

As does Branch et al., I define $M = (\bigoplus_{j=1}^{m} \beta_j)(P \otimes I_n)$, $\hat{y}_t = (y'_{1t}, y'_{2t}, \dots, y'_{mt})'$ and $\eta' = (\eta'_1, \dots, \eta'_m)'$ and where $\bigoplus_{j=1}^{m} \beta_j = diag(\beta_1, \beta_2, \dots, \beta_m)$.

This definition is important to the results formed by Branch et al. as it provides the exact condition needed for a unique uniformly bounded solution. That is, the eigenvalues of $M = (\bigoplus_{j=1}^{m} \beta_j)(P \otimes I_n)$, must lie inside the unit circle. A simple condition labeled, Conditionally Linear

[1] For the full form of the model refer to Appendix A.

Determinacy Condition (CLDC), provides the link to a regime dependent equilibria (RDE) through the MSV-solution.

Equation (5) corresponds to a multivariate linear rational expectations model but more importantly to a MSV-solution. The stacked system with well-known solutions characterized by McCallum (1983), is an important link for Branch et al. which defines expectational stability. Their analysis reveals the conditions which provides a uniformly bounded solution to the stacked system. Whereas this paper uses the MSV-solution result to form the conditions needed to provide a MSS-solution to equation (5).

THE MINIMUM STATE VARIABLE SOLUTION

The solution of the reduced form multivariate linear rational expectations model can be compared to the minimal state variable solution proposed by McCallum. Davig and Leeper offer a MSV-solution of the form,

$$y_t = \mathrm{B}(s_t)r_t.$$

Branch et. al. identify the solution of this form as a *Regime Dependent Equilibria* (RDE) and come to the conclusion the MSV is a RDE. They believe that the model, in this form can be solved using the techniques from Blanchard and Khan (1980). Standard techniques dictate that if the rational expectations equilibrium is unique the solution is determinate while indeterminate if there are multiple equilibria. Branch et al. reveal that their determinate condition is integral to one of their primary tenants: the characteristic root conditions which governs stability, the CLDC, is analogous to the Long-Run Taylor Principle created by Davig and Leeper (2007).

In their simple framework, Branch et. al. are able to create the conditions necessary to insure the stability of the model. They corroborate the findings from Farmer, Waggoner, and Zha which identifies the possibility of multiple equilibria occurring because of the positive feedback occurring from regime changes.

MEAN-SQUARE STABILITY

Stability concepts do not vary too much in the literature. Branch et al., as well as Davig and Leeper, use a familiar form of stability which relies on bounded stability. Farmer et al. are among the firsts to propose an alternative idea of stability, requiring the first and second moments of a stochastic process to be finite. I take up Farmer et al. and Cho's (2014) arguments for using MSS, which include 1) the ability to characterize a large set of stochastic processes that are covariance stationary, 2) relevant macroeconomic literature generally assumes unbounded, covariance stationary stochastic processes like normally distributed shocks, and 3) unlike the boundedness criteria for determinacy, the conditions for MSS in MSRE models has very tangible, immediate applications for analysis. Below, I outline the concept of mean-square stability using the work of both Cho and Farmer et al..

In my paper, I propose the use of an alternative method for analyzing stability in this class of models, Markov Switching Rational Expectations models (MSRE). By using Mean Square Stability (MSS) conditions models that may have positive regime feedback can be correctly analyzed for stability while using the criteria of a MSV solution. This is imperative considering the class of solutions the MSV satisfies, including adaptive learning.

Farmer et al. begin by adopting the indeterminate solution method written by Lubik and Schorfheide (2003, 2004) where they use a combination of the MSV-solution and a first-order moving average component. It can be written as,

$$y_t = Gu_t + w_t$$

$$w_t = \Lambda w_{t-1} + V\eta_t.$$

Some important considerations are that the shock term, η_t, is stable, zero-mean, and is also a non-fundamental disturbance that may or may not be correlated with the shock term u_t. η_t is also k dimensional, where k is the number of explosive eigenvalues. It follows that Λ is an *n x n* matrix of rank *k,* which can be written in the following form,

$$\Lambda = V\Phi V'.$$

Lubik and Schorfheide, by writing the solution in this form, have essentially removed any doubt from the question if there is a unique determinate solution. The question that is now posed from this form, is if this is a stable stochastic process. They use this methodology to show that monetary policy in the U.S. during the 1960s and '70s produced indeterminate results by not satisfying the well understood Taylor principle.[2]

This paper diverges from the canonical literature regarding stability in adaptive learning, since the concept of stability that I employ is mean-square stability rather than bounded stability. The formal definitions of both MSS and bounded stability are as follows,

Definition 1 *An $n\,x\,1$ stochastic process y_t is mean square stable (MSS) if there exists an n x 1 vector $\bar{y}$ and an n x n matrix Q such that $\lim_{t\to\infty}(E[y_t]-\bar{y}) = 0_{n\,x\,1}$ and $\lim_{t\to\infty}(E[y_t y_t{}']-Q) = 0_{n\,x\,n}$.*

Definition 2 *An n-dimensional process y_t is bounded if there exists a real number N such that $\|y_t\| < N, for\ all\ t.$*

where $\|*\|$ is a well-defined norm. It's important to note that if the process y_t is MSS it follows that it is also boundedly stable. For linear systems, these two concepts are identical for determining uniqueness of the equilibrium but in Markov-switching models, these two ideas are not the same and an economist must choose between the two. For a bounded process to be a tractable within this type of model, all possible products of the coefficient matrices must have characteristic roots inside the unit circle. No known conditions exist at this time for analyzing a bounded process.

Definition 3 *The stochastic reduced form, state dependent model is said to be determinate if there exists a stable fundamental solution, and there is no association of a stable non-fundamental solution, the stochastic component w_t with the fundamental solution.*

Intuitively definition 3 proposes that a stable solution can exists if a stable bubble fails to exist. Agents must not be able to create a self-fulfilling stable equilibrium.

[2] For a practical example please refer to pg. 11 from Farmer et al. (2009)

The following lemma characterizes the solutions to the stacked system, regardless of MSS and the conditions for existence a MSV equilibrium.

Lemma 1 *Any solution to equation the stacked system can be written in the following way:*

$$y_t = G_{s_t}u_t + w_t,$$

$$w_t = \Lambda_{s_{t-1},s_t}w_{t-1} + V_{s_t}V_{s_t}'\gamma_t,$$

where V_{s_t} *is an* $n\,x\,k_{s_t}$ *matrix with orthonormal column and* $0 \le k_{s_t} \le n, \gamma_t$ *is an arbitrary n-dimensional shock process s.t.* $E_{t-1}[V_{s_t}V_{s_t}'\gamma_t] = 0, \Lambda_{s_{t-1},s_t}$ *is an n x n matrix of the form* $V_{s_t}\Phi_{s_{t-1},s_t}V_{s_{t-1}}'$ *for some* $k_{s_t}\ x\ k_{s_{t-1}}$ *matrix* Φ_{s_{t-1},s_t} *s.t.*

$$\Gamma_i V_i = \sum_{j=1}^{h} p_{ij}V_j\,\Phi_{ij}\ for\ 1 \le i \le h,$$

And $G_{s_t}u_t$ *is the minimum-state variable (MSV) solution with* G_{s_t} *representing the conformable coefficient matrix from the forward looking component of the stacked system.*

Proof *See Appendix.*

Lemma 1, originally from Farmer et al. (2009), provides an important result for the stacked system. The form of the solution is similar to the Lubik-Schorfheide representation of linear systems. Their representation comprises both a fundamental and non-fundamental component, the MSV solution and the moving average vector respectively. The result indicates that the moving average component is indeed a Markov switching system, and thus can be subjected to the necessary and sufficient conditions for determinacy.

DETERMINACY USING MEAN-SQUARE STABILITY

Again following the framework provided by Lubik and Schorfheide, Cho considers a solution which includes both a fundamental and non-fundamental (sunspot) solution. The fundamental component of a Linear Rational Expectations model takes on the form,

$$x_t = [\Omega(s_t)x_{t-1} + \Gamma(s_t)z_t] + w_t,$$

where w_t, is a non-fundamental component of the form,

$$w_t = E_t[F(s_t, s_{t+1})]w_{t+1}.$$

It is important to note that the coefficient matrices in both the fundamental and non-fundamental components must satisfy the following conditions for each regime, $s_t \; and \; s_{t+1}$,

$$\Omega(s_t) = \{I_n - E_t[A(s_t, s_{t+1})\Omega(s_{t+1})]\}^{-1}B(s_t),$$

$$\Gamma(s_t) = \{I_n - E_t[A(s_t, s_{t+1})\Omega(s_{t+1})]\}^{-1}C(s_t),$$

$$F(s_t, s_{t+1}) = \{I_n - E_t[A(s_t, s_{t+1})\Omega(s_{t+1})]\}^{-1}A(s_t, s_{t+1}).$$

Cho and Farmer et al. emphasize the complexity of determining the stability of the non-fundamental components. Cho proves that the non-fundamental solution can be written in the following way:

$$w_{t+1} = \Lambda(s_t, s_{t+1})w_t + V(s_{t+1})V(s_{t+1})'\eta_{t+1},$$

where $V(s_t)$ can be thought of as a $nxk(s_t)$ matrix with orthonormal columns, $0 \le k(s_t) \le n \; and \; k(s_t) > 0$ for some s_t. η_t is an arbitrary $n \; x \; 1$ innovation s.t. $E_t[V(s_{t+1})V(s_{t+1})'\eta_{t+1} = 0_{n \, x \, 1}$, $\Lambda(s_t, s_{t+1}) = V(s_{t+1})\Phi(s_t, s_{t+1})V(s_t)'$ for some $k(s_{t+1})$ x $k(s_t)$ matrix $\Phi(s_t, s_{t+1}) \; s.t.$

$$V_i = \sum_{j=1}^{S} p_{ij} \, F_{ij}V_j\Phi_{ij}, for \; 1 \le i \le S,$$

where $V_i = V(s_t = i), \Phi_{ij} = \Phi(s_t = i, s_{t+1} = j) \; and \; F_{ij} = F(s_t = i, s_{t+1} = j)$.

Consider the following stochastic process:

$$y_{t+1} = G(s_t, s_{t+1})y_t + H(s_{t+1})\eta_{t+1},$$

where y_{t+1} is a $n \; x \; 1$ vector, $G(s_t, s_{t+1})$ and $H(s_{t+1})$ are $n \; x \; n \; and \; n \; x \; l$ matrices respectively. η_{t+1} is an arbitrary $l \; x \; 1$ vector assumed to be mean-square stable. In order to work with a model similar to the MSRE models, Cho transforms the stochastic equation so that both *G* and *H* depend on s_t, while the disturbance term is measured at time *t*. To assess MSS, we will focus on the homogenous component of, the stochastic process. Let $G_{ij} = G(s_t = i, s_{t+1} = j)$. The following matrices can be defined:

$$\Psi_G = p_{ij}G_{ij} = \begin{bmatrix} p_{11}G_{11} & \cdots & p_{1S}G_{1S} \\ \cdots & \cdots & \cdots \\ p_{S1}G_{S1} & \cdots & p_{SS}G_{SS} \end{bmatrix}, \overline{\Psi}_G = p_{ji}G_{ji} = \begin{bmatrix} p_{11}G_{11} & \cdots & p_{S1}G_{S1} \\ \cdots & \cdots & \cdots \\ p_{1S}G_{1S} & \cdots & p_{SS}G_{SS} \end{bmatrix},$$

$$\Psi_{G\otimes G} = p_{ij}G_{ij} \otimes G_{ij} = \begin{bmatrix} p_{11}G_{11} \otimes G_{11} & \cdots & p_{1S}G_{1S} \otimes G_{1S} \\ \cdots & \cdots & \cdots \\ p_{S1}G_{S1} \otimes G_{S1} & \cdots & p_{SS}G_{SS} \otimes G_{SS} \end{bmatrix},$$

$$\overline{\Psi}_{G\otimes G} = p_{ji}G_{ji} \otimes G_{ji} = \begin{bmatrix} p_{11}G_{11} \otimes G_{11} & \cdots & p_{S1}G_{S1} \otimes G_{S1} \\ \cdots & \cdots & \cdots \\ p_{1S}G_{1S} \otimes G_{1S} & \cdots & p_{SS}G_{SS} \otimes G_{SS} \end{bmatrix},$$

where $\otimes$ denotes the Kronecker product.

Definition 4 *The spectral radius of an n x n matrix M is defines as* $r_\sigma(M) = \max_{1\le i\le n}(|\lambda_i|)$, *where* $\lambda_1, \dots, \lambda_n$ *are the eigenvalues of M.*

The engineering literature that defines the determinacy and indeterminacy of MSS, proposes the following theorem.[3]

Theorem 1 *The stochastic process,* y_{t+1}, *is mean-square stable if and only if the spectral radius of* $\overline{\Psi}_{G\otimes G}$ *is less than one. That is* $r_\sigma(\overline{\Psi}_{G\otimes G}) < 1$.

Adapting Theorem 1 to the MSRE models requires the fundamental solution have the same form as stated above, again assuming the vector of shock terms is already mean-square stable. That is $x_t = \Omega(s_t)x_{t-1} + \Gamma(s_t)z_t$ is mean square stable if and only if

$$r_\sigma(\overline{\Psi}_{\Omega\otimes\Omega}) < 1,$$

As for the non-fundamental component $w_t = \Lambda(s_{t-1}, s_t)w_{t-1} + V(s_t)V(s_t)'\eta_t$, w_t is mean-square stable if and only if

$$r_\sigma(\overline{\Psi}_{\Lambda\otimes\Lambda}) < 1.$$

Again, we assume the sunspot error term is white noise. Thus we get the following theorem.

[3] For further information regarding the notation and creation of the MSS matrices, please refer to Costa (2005).

Theorem 2 *The stochastic process, with both fundamental and non-fundamental components is found to be uniquely determinate under mean-squared stability if and only if* $r_\sigma(\overline{\Psi}_{\Omega\otimes\Omega}) < 1$, *and* $r_\sigma(\overline{\Psi}_{\Lambda\otimes\Lambda}) < 1$.

Proof. *See Definition 1.*

Theorem 2 says that the non-fundamental component must satisfy the MSS conditions. If this occurs, there is no sunspot equilibria competing to govern the dynamics of the regime switching system.

ADAPTIVE LEARNING AND E-STABILITY

Current monetary policy can be confusing even for a seasoned economist. As the Federal Reserve is transitioning to a new chair president, speculation in regards to policy changes most likely will increase. Speculation regarding policy is often seen as a driver for investment volatility. Recent Fed policy aims to create a greater transparency in policy expectation by announcing current and future plans. But it is important to note that the inner workings of policy decisions are very complex and hinge on the understanding of key, dynamic relationships. As macroeconomics moves forward as a science, previous ideas of these relationships need to be edited in order to accommodate new vision. One such idea that has endured is Rational Expectations (RE).

From its inception and then application, RE has disseminated into macroeconomics thus becoming the standard for all forward-looking models. As noted by Branch, Davig and McGough (2013), "A given forward-looking macroeconomic model may admit classes of rational expectations equilibria that differ in terms of the set of state variables that agents use when forming expectations."

A direct response to these criticisms of RE is Adaptive learning (AL), which allows agents to have close to rational expectations. This allows agents to have reasonable idea regarding the system which motivates the economy but fails at knowing the exact parameter values. Thus agents learn about these parameters over time. Over the last 20 years, the methodology behind AL has been thoroughly researched, starting with the idea of bounded rationality emphasized by Sargent (1993, 1999), and then the didactic work from Evans and Honkapohja (1999, 2001).

In their book regarding the application and methodology behind AL, Evans and Honkapohja (2001) explore the use of Markov switching parameters and the effects of sunspot equilibrium. What their book and recent literature lacks though is a comprehensive treatment of a more complex international macroeconomic model. My research begins to fill that void; it pieces together a model which explores issues previously ignored, such as exchange rate fluctuations.

MULTIVARIATE ADAPTIVE LEARNING FRAMEWORK

This section builds up the framework necessary for providing a link between the MSV-solution, mean-squared stability, and E-Stability. I assume that agents are able to observe the transition probabilities of each state, s_t but do not observe the vector of endogenous variables, y_t. These assumptions feel plausible considering agents can fully observe the transition of leadership in the monetary authority but cannot fully observe the economic variables driving the model. As Branch et al. note, this is standard within the AL literature, which assumes agents only observe contemporaneous exogenous variables but not endogenous ones.[4]

Using the reduced form model, equation (8), agents observe the set of equations governing the dynamics of this system and

$$y_t = ME_t^* y_{t+1} + \eta r_t,$$

where E_t^* is an ambiguous expectational process, r_t is defines by the form, $r_t = \rho r_{t-1} + \nu_t$. It is further assumed that $0 < \rho < 1$ and ν_t is a white noise process. Providing the determinacy of *M*, under RE there exists a unique stable equilibrium of the form $y_t = \mathrm{B} r_t$.

[4] Branch et al. note that this assumption only appears to be strong but is in fact also assumed in the rational expectations literature. It is because the literature on adaptive learning strives to replicate the results rational expectations do researches continue to use this assumption.

Under learning, agents build their forecasting model without the knowledge of parameter values, this is referred to the Perceived Law of Motions (PLM),

$$y_t = A + Br_t.$$

Following the MSV-solution setup, in this form, the conformable, well-behaved, parameter matrices *A* and *B* capture the perceived relationship between the endogenous vector of variables, y_t and the error term, r_t. Adaptive learning provides the econometricians the opportunity to estimate matrices *A* and *B* through different learning algorithms, e.g. recursive least squares.

Agents learn by using the data available to them up until time *t*, and forecast using their PLM,

$$E_t^* y_{t+1} = A_{t-1} + B_{t-1}\rho r_t.$$

Plugging their forecast back into the reduced form yields the Actual Law of Motions (ALM),

$$y_t = MA_{t-1} + (MB_{t-1}\rho + \eta)r_t.$$

Assuming that agents know the evolution of r_t, it becomes clear how forecasts of endogenous variables, in time *t*, depend solely on the perception of determined last period, A_{t-1} and B_{t-1}. When new information becomes available, agents update their perceptions, A_t and B_t to make new forecasts on the endogenous variables. This process continues until the perceived parameter matrices either diverge or coincide with the rational expectations equilibrium. When they coincide the model is said to be stable under learning; that is $(A_t, B_t) \rightarrow (0, \mathrm{B})$ almost surely. To illustrate this condition, known as E-Stability, we look toward the generic beliefs of the agents, (*A, B*). The ALM defines a function, or map, where$T\colon \mathbb{R}^2 \rightarrow \mathbb{R}^2$.

$$T(A, B) = (MA, MB\rho + \eta).$$

To find an equilibrium from the T-map, one must only look for the fixed point in the map. This point identifies the rational expectations equilibrium (REE). E-Stability is then determined by "moving" locally around this fixed point and observing the asymptotic convergence. If the fixed point of the ordinary differential equation (o.d.e) is locally asymptotically stable then the REE is said to also be E-Stable,

$$\frac{d(A,B)}{d\tau} = T(A,B) - (A,B).$$

Intuitively, the o.d.e. is thought of as the forecasting error produced by the agents. As Branch et al. put it, “if the resting point of the o.d.e. is stable then adjusting parameters in the direction indicated by the forecast error will lead the parameters toward the REE.”

Fortunately, this is fairly easy to compute. Local asymptotic stability can be assessed using the eigenvalues of the Jacobian matrix *DT.* E-Stability within state-contingent models arises when the eigenvalues have real parts of modulus one. Branch et al. bring readers to their main tenet: The derivative of the Jacobian matrix, which governs E-stability, coincides with the matrix that satisfies the Conditional Linear Determinacy Condition. This connection between MSV and E-stability, especially within a regime-dependent equilibrium, is paramount to their results.

E-STABILITY AND MEAN-SQUARE STABILITY

The connection between mean-square stability and E-Stability relies heavily on the CLDC created by Branch et al. That is, I prove if a solution, in the form of a MSV also satisfies the conditions for MSS it is E-stable through the CLDC.

In order to insure that the MSS solution coincides with the MSV solution, Farmer et al. provide the following corollary to Lemma 1.

Corollary 1 *Let* $0 \leq k_i \leq n$. *Consider the problem of choosing* $n\ x\ k_i$ *matricies* V_i *and* $k_j\ x\ k_i$ *matricies* Φ_{ij} *such that* $r_\sigma\left(M_1(\Phi_{ij})\right)$ *is minimized subject to the constraints* $\Gamma_i V_i = \sum_{j=1}^{h} p_{ij} V_j \Phi_{ij}$ *and* $V_i' V_i = I_{k_i}$.

If, for all possible choices of $\{k_1, \cdots, k_h\}$*, not all zero, the minimum value of the* $r_\sigma\left(M_1(\Phi_{ij})\right)$ *is greater than or equal to one, there will be only one mean-square stable solution to the model. This solution is the MSV solution. Otherwise there will be multiple solutions to the stacked system.*

Proof *See Appendix.*

Indicated by Farmer et al., MSS implies the bounded stability conditions found in most economic analysis including that of Branch et al.. That is, if the necessary and sufficient conditions are met for MSS then they also imply the existence of a unique uniformly bounded solution. Farmer et al. do not offer any formal proof as they cite it is a widely known result from the engineering literature. Thus, a MSS solution satisfies the condition of a CLDC. This is formalized in Proposition 1 below.

First, I must establish the central results from Branch et al. with Lemma 2 and 3.

Lemma 2 *If the CLDC holds then there is a unique RDE that corresponds to the MSV-Solution.*

Proof *See Appendix.*

Lemma 2 describes the realization that any RDE solves the stacked system. This follows from the proof shown in the appendix. Again the fault in their analysis stems from the weak conditions for creating the CLDC. Recall, that the CLDC only requires the characteristic roots of the coefficient matrix in the stacked system to be within the unit circle. With that noted, Lemma 3 provides the connection for E-stability.

Lemma 3 *If the CLDC holds, then the unique RDE is E-stable.*

Proof *See Appendix.*

Proposition 1 *A unique MSV-solution which satisfies the conditions for mean-square stability is also E-Stable.*

Proof *The proof follows as such.*

Corollary 1 identifies the conditions necessary for a unique MSS solution that is the MSV-solution.

Since the necessary and sufficient conditions for MSS imply bounded stability in the sense of an RDE. It follows then, that a MSS solution is an RDE and thus satisfies the CLDC. The remainder of the proof then comes from Lemmas 2 and 3.

QED■

Since the CLDC holds under MSS-solutions, a MSS-solution is E-stable. Proposition 1 is the main result of this paper. By linking the methodology found in MSS economists can move forward from rational expectation model to ones which include adaptive learning.

Both lemmas 2 and 3 come from Branch et al. to show the existence of an E-stable solution. They succinctly make the connection between the MSV-solution and the regime dependent equilibrium through the CLDC.

CAGAN MODEL IN AN OPEN ECONOMY

In order to further understand the underlying economic ideas guiding a stable forward solution, I present an economic application: a small open economy model.

Exchange market pressure models [EMP] employ a non-linear Markov switching parameters, similar to the Cagan model presented earlier. Kumah (2011) argue that EMP models have been proven to be empirically better than more traditional VAR models when exploring exchange rate differentials. This result is important because we can analyze the active versus passive reaction of the monetary authority when there is excess supply or demand of the currency. This section explores the EMP model and how stability occurs when mean-square stability conditions are introduced. I begin with the linear, expectational difference equation

$$m_t^d = (e_t + p_t^*) + \alpha y_t - \beta(i_t^* + E_t \Delta e_{t+1}) + v_t,$$

where α, is the income elasticity of money, β is the interest semi-elasticity of money, and the shock term v_t is an i.i.d. term representing unanticipated money demand shock. Furthermore, p_t^* represents foreign prices and i_t^* is the foreign interest rate. It is the standard assumption in this model that both the UIP and PPP conditions hold.

Domestic money supply can be expressed additively as the combination of the domestic credit and foreign reserves,

$$m_t = d_r + r_t.$$

The monetary authority intervenes by selling and purchasing foreign exchange according to the rule,

$$\Delta r_t = -\chi(s_t)\Delta e_t.$$

At this point, the monetary authority's decision creates a non-linearity in the model since their sensitivity parameter, χ, is dependent on the state (regime), s_t. Under this general framework, one could assume two possible monetary policy regimes: active and passive. The question now becomes, given standard values for the remaining parameters, what combinations of active and passive policy are supported by mean-square stability? The answer to this question allows researchers to compare the theoretical range of stable values to those that empirically estimate the sensitivity of the central banks. Ultimately, one could conclude whether or not a central bank was working toward stability.

After substituting in the domestic money supply and the monetary authority's intervention, while taking the first difference, we find that

$$\Delta e_t = \frac{1}{1+\beta+\chi(s_t)}(-\Delta p_t^* - \alpha\Delta y_t + \beta\Delta i_t^* + \beta E_t\Delta e_{t+1} - \Delta v_t + \Delta d_t).$$

From this form of the model we can conclude that the exchange rate dynamics are consistent with theoretical literature.[5] Furthermore it becomes apparent that the monetary policy parameter is very important for the determination of the exchange rate. For instance, as $\lim_{\chi(s_t)\to\pm\infty} \Delta e_t = 0$, the monetary authority would be holding the exchange rates fixed. Moreover, as $\chi(s_t) \to 0$, the exchange rate is allowed to freely float given changes in economic fundamentals. Intermediate policy can be summarized in a simple expression, that is when $0 < \chi(s_t) < \infty$. In this case, the monetary authority mitigates appreciations (depreciations) by purchasing (selling) foreign exchange. When the central bank chooses $-(1+\beta) < \chi(s_t) < 0$ or $\chi(s_t) < -(1+\beta)$ the monetary authority is either magnifying exchange rate changes or leaning against the wind.

[5] See Dornbusch 1976.

To simplify the model I assume that foreign prices, domestic output and foreign interest rates remain constant, $\Delta p_t^* = \Delta y_t = \Delta i_t^* = 0$. The model further reduces to the non-linear, Markov-switching regime form developed earlier,

$$\Delta e_t = \frac{1}{1 + \beta + \chi(s_t)} (\beta E_t \Delta e_{t+1} - \Delta v_t + \Delta d_t).$$

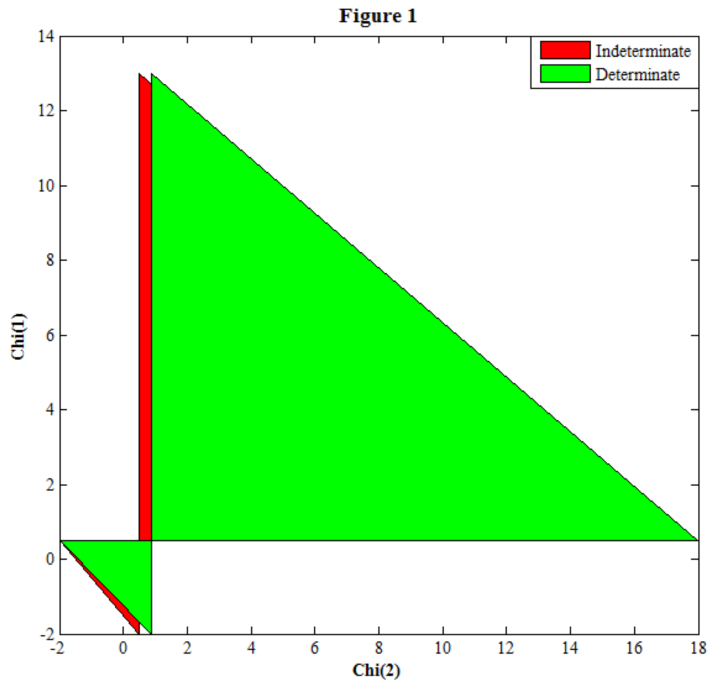
DETERMINACY UNDER CAGAN MODEL

By first assuming the interest semi-elasticity of money to be one and that the central bank has a regime probability matrix defined as

$$p_{ij} = \begin{bmatrix} .8 & .2 \\ .1 & .9 \end{bmatrix},$$

For a specific parameterization, I find the region of determinacy and indeterminacy below in Figure 1. Using the economic intuition derived about $\chi(s_t)$, the parameter region is bound by the domain [-2,18] and range [-2,14]. By choosing this set of values, I allow for all possible responses from the central bank. It should be noted, that as the space increases, so does the determinacy region.

Upon inspection of Figure 1, it becomes clear that the forward solution only exists when the monetary authority follows an intermediate policy for both active and passive policy. Leaning against the wind and magnification of exchange rate changes are not considered stable solutions under mean-square stability conditions. Moreover, even a small intermediate response, in magnitude, is not enough to keep the forward solution for the exchange rate stable. One can infer then that even an aggressive active approach needs to be met with a somewhat large passive approach. This can be attributed to agents expecting the possibility of regime change.



Furthermore, recall that according to theorem 2, stability conditions are met when $r_\sigma(\overline{\Psi}_{\Omega\otimes\Omega}) < 1,$ *and* $r_\sigma(\overline{\Psi}_{\Lambda\otimes\Lambda}) < 1,$ that is when the fundamental solution exists when the non-fundamental solution does not. Given the parameterization where $\chi(1) = 2\ and\ \chi(2) = 1.5,$ I find that

$$r_\sigma(\overline{\Psi}_{\Omega\otimes\Omega}) = 0\ and\ r_\sigma(\overline{\Psi}_{\Lambda\otimes\Lambda}) = 0.07,$$

which indicates that the forward solution exists without a bubble solution thus rendering this Markov-switching regime as stable.

EMPIRICAL TEST OF DETERMINACY

In order to test the stability of the model empirically, I analyze exchange rate data between the US, Germany, and Japan from the late 1970's through to the early 1990's. Using the Plaza and Louvre accords as natural regime changes in exchange rate policy, I am able to use the monetary policy equation presented in the hyperinflation model in order to estimate the sensitivity of the central bank to changing exchange rates. The regression that I estimate is below:

$$\Delta r = -\chi \Delta e.$$

For this empirical work, I turn to the FRED database maintained by the Federal Reserve of St. Louis. The three time-series data sets I employ are the: German Deutsche Mark to U.S. Dollar exchange rate, Japanese Yen to U.S. Dollar Exchange rate, and the Federal Reserve's holdings of Japanese Yen. The final data set was procured from the Bundesbank, which captures Germany's central bank holdings of U.S. dollars. Daily exchange rates were averaged to obtain monthly figures, while all data were not seasonally adjusted. Lastly, the regression was run with the constant term restricted to be zero.

The regime changes in monetary and exchange rate policy during the Louvre and Plaza accords, offered a natural experiment for testing the stability of each countries' currency market interventions. Empirical support shows that this time period of U.S. monetary policy exhibited at least one structural break during 1985, which coincides economically with the signing of the accords from these three countries. The regression results are displayed in Table 1 below.

Pre-Plaza accord

Using roughly nine years of monthly data prior to the signing of the Plaza accord, I find that as the exchange rate appreciates, the central bank responded with an increase in foreign reserves. A 1% appreciation in the exchange rate, i.e. the U.S. dollar (USD) appreciating relative the Yen, results in a significant decrease of 2.8% of Yen held by the Federal Reserve. By supplying more Yen and removing USD from circulation, the Fed was exacerbating the problem of currency appreciation.

From the perspective of Germany, the period of time before the Plaza accord was signed, the Bundesbank had been decreasing their holdings of U.S. dollars by 0.75% when experiencing a depreciation of the DEM relative to the USD, a significant reduction. During this time, the USD had been appreciating against both the Yen and the Deutsche Mark (DEM), which makes the actions taken by the monetary authority of Germany plausible.

Unfortunately, the currency market does not work bilaterally and a simple open economy model obviously does not include the complexity of each countries monetary policy, therefore I do not find it fruitful to make comparison between the magnitudes of responses from each country. What I can observe though is the general reaction from each country and how monetary policy changes affected the stability of the model.

Plaza accord

After the plaza accord was signed, the countries of the US, UK, Japan, France, and West Germany agreed to depreciate the dollar relative to the yen and Deutsche Mark by intervening in the currency market., thus began the Federal Reserve's policy of USD depreciation. As less Yen and more dollars become available the interest rates rises and falls respectively for each country. This results in a shift of investment from the United States to Japan as agents looking to loan money seek the higher interest rate. This floods the market with dollars and appreciates the yen while depreciating the dollar.

As expected, the direction of the sensitivity parameter now reveals that monetary policy was depreciating the USD. Reserves of the Yen increased significantly by 1.9% as the exchange rate appreciated by 1%. This implies an aggressive strategy to change the level of foreign reserves. Intuitively, the planned intervention into the currency market would signal a regime change in the monetary policy rule.

Germany continued their monetary policy by decreasing their holding of USD as DEM appreciated against the USD. Although, their exchange rate sensitivity isn't significantly different from zero it signals a strategy by the Bundesbank to allow their exchange rate to float freely. It seems that the Bundesbank might have been relinquishing some control of the exchange rate and the currency market and allowing the policy of the Federal Reserve to regulate the market.

Louvre accord and beyond

Given the overwhelming success of the Plaza accord, the same nations agreed to now appreciate the dollar by signing the Louvre accord. Again the nations would intervene strategically in the currency market to appreciate the dollar against the Yen and Deutsche Mark. We would expect that the direction of the Federal Reserve's policy parameter to switch, signaling a change in policy to appreciate the dollar. This is exactly what I find. The change in exchange rate parameter for the United States switches from negative to positive while the magnitude increases greatly and remains significant. This particular intensity of intervention was followed for approximately 18 months. In the months and years after the Louvre accord, the Federal Reserve's policy toward appreciating the dollar continued but not at a rate significantly different from zero.

During the Louvre accord, the Bundesbank responded to increases of the exchange rate by increasing their reserve of USD by a significant 1.01%. By increasing their reserves of USD, the Bundesbank actively participated in appreciating the USD. In the time after the Louvre accord, the Bundesbank continued their acquisition of the USD to appreciate the USD against the DEM. The sensitivity of the exchange rate decreased to 0.35 when the exchange rate appreciated by 1%, still a significant amount.

Table 1. The Bundesbank's Holdings of USD

	Pre-Plaza Accord	Plaza Accord	Louvre Accord	Post-Louvre Accord
χ	-0.749**	-.0282	-1.071*	-0.345*
	(0.165)	(0.416)	(0.499)	(0.160)
n	176	17	17	125
R-Squared	0.10	0.03	0.22	0.04

Note: Regressions were run with a suppressed constant term. * refers to estimates being significant at the 95% confidence level, while ** refers to significance at the 99% confidence level. Furthermore, robust standard errors were created but revealed no qualitative difference from the reported coefficients.

Table 2. The Federal Reserve's Holdings of YEN

	Pre-Plaza Accord	Plaza Accord	Louvre Accord	Post-Louvre Accord
χ	2.837*	-1.922**	7.574*	1.18
	(1.367)	(0.614)	(3.421)	(0.769)
n	83	17	17	113
R-Squared	0.05	0.38	0.23	0.02

Note: Regressions were run with a suppressed constant term. * refers to estimates being significant at the 95% confidence level, while ** refers to significance at the 99% confidence level. Furthermore, robust standard errors were created but revealed no qualitative difference from the reported coefficients.

Stability Results

Using the necessary and sufficient conditions of mean-square stability, I analyze each regime change to determine whether the monetary policy was stable during the change.

Condition for stability: $r_\sigma(\overline{\Psi}_{\Omega\otimes\Omega}) < 1$, *and* $r_\sigma(\overline{\Psi}_{\Lambda\otimes\Lambda}) < 1$

Table 3. Regime changes for the Federal Reserve

	Pre-Plaza to Plaza Accord	Plaza to Louvre Accord	Louvre to Post-Louvre Accord
$r_\sigma(\overline{\Psi}_{\Omega\otimes\Omega})$	*0*	*0*	*0*
$r_\sigma(\overline{\Psi}_{\Lambda\otimes\Lambda})$	*147.92**	*131.49**	*0.09*

Note: * refers to non-stable monetary policy during regime switches

By applying the same parameterization as earlier, it becomes clear that the exchange rate policy instituted by the Federal Reserve was only stable after the intervention ended in the Post-Louvre accord era.

Table 4. Regime changes for the Bundesbank

	Pre-Plaza to Plaza Accord	Plaza to Louvre Accord	Louvre to Post-Louvre Accord
$r_\sigma(\overline{\Psi}_{\Omega\otimes\Omega})$	*0*	*0*	*0*
$r_\sigma(\overline{\Psi}_{\Lambda\otimes\Lambda})$	*0.52*	*1.05**	*0.94*

Note: * refers to non-stable monetary policy during regime switches

As for the regime positions of the Bundesbank, only during the Plaza to Louvre accord was the exchange rate policy not stable. Corroborating the evidence found in the stability results for the Federal Reserve, that the monetary policy during the Plaza accord was not sustainable.

CONCLUSIONS AND DISCUSSION

In this paper, I am able to advance the literature on non-linear, regime shifting models by studying the mean-square stability condition that also apply to the stability under adaptive learning. As defined by Branch et al., satisfying the mean-square stability conditions ensures the existence of a unique regime dependent equilibria. In turn this further indicates that this system is E-Stable under adaptive learning. This finding is the central tenant of this paper. The link derived between mean-square stability and E-Stability is important since the only previous attempts to find regime stable conditions fails to create tractable outcomes. As a result of this research, economists are now able to include the concept of bounded rationality even in the face of noisy regime feedback.

This paper looks at one application of the work, a univariate exchange rate model inspired by the work of Cagan. I find two main results. The first is that only intermediate active and passive responses by the monetary authority are found to be stable. The second finding is that the starting parameterization matters. From the first result, I conclude that given the possibility of regime switching, central banks no longer have the possibility of leaning against the wind or amplifying the change in the exchange rate. From the second, it is clear that the many combinations of parameters exist which can induce three outcomes: determinate, indeterminacy due to the existence of bubble solution, and no solution.

By testing the hyperinflation model empirically, using data that spans the 1970's-1990's, I am able to estimate and test the stability of the coefficients during each regime. What I find is that the response of the Federal Reserve to changing exchange rates dominated that of the Bundesbank in Germany. Moreover, the regime changes during that time produces unstable model dynamics and were only corrected when active intervention into the currency markets began to wane in the early part of the 1990's. Further testing of participating countries, like Japan, could reveal an outcome which helps explains their economic hardships during the 1990's.

Further research would be necessary in order to incorporate the possibility of a two-country, Markov-switching regime. Since the theoretical framework presented in this paper is valid in a multivariate framework a new-Keynesian model could explore the interaction between monetary authorities as they pass between active and passive regimes in order to both manipulate the exchange rate.

CHAPTER 2 "CONVERGENCE AND E-STABILITY OF A REGIME SWITCHING MONETARY MODEL OF EXCHANGE RATE"

INTRODUCTION

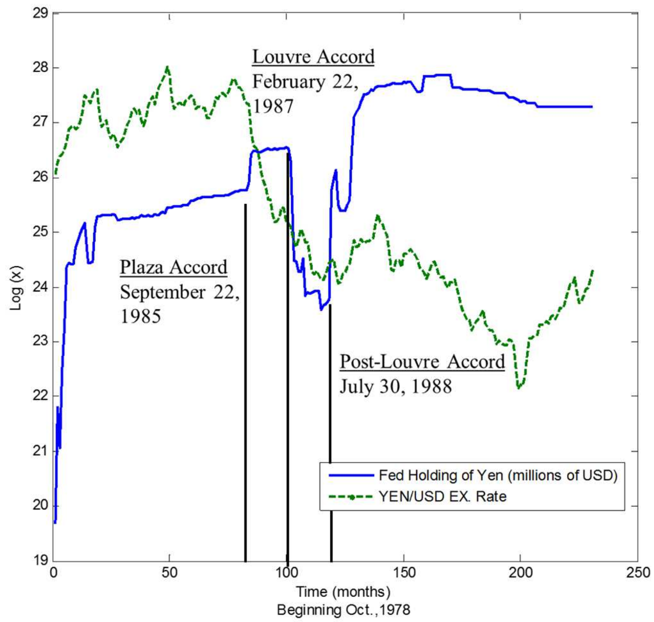
Predicting exchange rate values was recently resurrected beginning with Nelson Mark (1995) when he showed that model predictions of the exchange rate can outperform random walk forecasts. This seminal work paved the way for future research into the forecasting of exchange rates through a variety of macro-econometric models. This paper builds upon more recent literature in adaptive learning in order to understand the responses by economic agents to specific regime changes in monetary fundamentals. Unlike rational expectations, agents in the model are assumed to know certain parameter values and the underlying model construction but are required to estimate certain deep parameters, which govern convergence and stability. By assuming that agents are bounded in their rationality, this body of literature is able to incorporate misspecification, sub-optimal decision making, and systematic errors into the confines of traditional rational expectations analysis.[6] One of the key criticisms of rational expectations models of exchange rate determination has been the systemic under-prediction of the volatility in exchange rate movements by economic fundamentals. This paper aims to reevaluate the volatility produced by exchange rate movements under the assumption that agents use an adaptive learning approach to understand state dependent parameters. Moreover, this paper highlights the slow convergence to a rational expectations outcome which accompanies the empirical data. I am able to show that the convergence of the learned parameter to rational expectations was slower during periods after a regime change, thus creating an economic climate of policy uncertainty during the mid-1980's in the United States.

During the 1980's, the exchange rate between the United States and some of its global partners, especially West Germany and Japan, experienced a high rate of appreciation. Due in part to tightening monetary policy by Paul Volker's Federal Reserve, U.S. dollars (USD) became

[6] For a more extensive list regarding the qualitative comparisons between adaptive learning and rational expectations, I refer readers to Evans and Honkapohja (2001).

increasingly attractive. Between 1980 and 1983, the exchange rate appreciated by approximately 42%, in real terms, relative to an indexed global currency, and another 20% from 1984 to 1985 (Feldstein 1994). In order to disrupt this appreciation, the United States, Great Britain, West Germany, Japan, and France developed a currency market intervention schedule to begin depreciating the USD against their national currencies. This intervention, known as the Plaza accord, marked a specific policy regime change which was followed 17 months later by the Louvre accord. This follow up to the Plaza accord began a period of appreciating the USD against the other currencies. The policy was needed as the USD depreciation was approximately equal in magnitude to the sharp appreciation. The Louvre accord was positioned as a stabilization policy targeted to bring the exchange rate back in line with historic and competitive levels. This period from 1987 to 1990 marked a volatile yet trend stationary period in the exchange rate. Following this second market intervention the exchange rate was relatively allowed to free float against the remaining currencies. This paper explores the possibility that the appreciation leading up to the Plaza accord and the subsequent currency market intervention created an exchange rate adjustment which overshot its target due in part to the bounded rationality of economic agents. In the figure below, the drastic influence in the currency market can be seen as the Federal Reserve began to increase their holding of the YEN around the time of the Plaza accord and then feverishly unload YEN during the Louvre accord, to then finally accumulate YEN again as the exchange rate continued to depreciate.

Figure 1. Federal Reserve's holdings of YEN compared to the YEN/USD Ex. Rate



This paper also addresses the exchange rate bubble which was observed between 1983 and 1984 and subsequently began to burst in February 1985, depreciating by 13% before the Plaza meetings in September of the same year. Because of the gap in policy, many economists argued that exchange rate interventions had little to no effect on the actual fluctuations of the dollar and was instead dependent upon by the private market with little influence from the government at all (Feldstein 1986). Frankel, Bergsten, and Mussa (1994) provide an alternative rationale for the connection. The perception among market participants was that key monetary policy decision makers were more adamant at bringing down the appreciating exchange rate than were their predecessors. This led investors to anticipate the depreciation and thus sell their dollars today to insulate themselves from future losses. Under this assumption, agents expectations played a significant role in the in the depreciation of the exchange rate, even more so than actual policy. This can be seen in the second rationale presented by Frankel, et al. During the first quarter of 1985, the United States engaged in $659 million of currency market intervention with the Bundesbank of Germany and other nations selling approximately $10 billion in the foreign exchange market.[7] Given these interpretations of the bubble and the collapse before the Plaza accord, it appears that expectations played a significant role in the fluctuation of the exchange rate. Under rational expectations, agents should have full knowledge of the model structure and each policy parameter but because of exchange rate instability the assumption of rational expectations appears to be too strict. Even during times of stable policy, economists would be required to estimate parameter values econometrically. On the other hand, adaptive learning allows economic agents to face some limitation on the true knowledge of the economic climate. Adaptive learning will allow agents to assume the functional, reduced form of the model but will need to learn the parameter values through some method of least squares learning. By moving away from the rational expectations hypothesis and thus loosening the assumptions on expectations, this research provides evidence for the rationales presented by Frankel, et al. considering the market participants during the 1980's would not have known the direction and magnitude of the policy parameters.

In order to explore this possibility, this paper draws on the conclusions of Kim (2008) regarding the use of adaptive learning in monetary models of exchange rates. Kim finds that the use of adaptive learning in comparison to rational expectations and adaptive expectations dominates the forecasting of exchange rates over long time horizons. Furthermore, in his simulations, the use of bounded rationality accounts for the presence of exchange rate volatility above what is found within the monetary fundamentals. Moreover, it appears that the inclusions of adaptive learning also helps to explain the persistent deviations of the exchange rate away from the monetary fundamentals. Although Kim begins to posit the outcome of exchange rate behavior during regime switching, the analysis relies on the econometric analysis of two different regime periods or cohorts of data instead of focusing on agents learning the regime change through some updating algorithm. The lack of formal Markov switching dynamics leaves the

[7] Federal Reserve Bank of New York Quarterly Review (Spring 1985)

model unprepared for handling the complicated relationship which competing regimes present. This paper builds on the foundation set by Kim but includes the appropriate regime dynamics present in earlier papers by Reed (2014) and Branch, Davig and McGough (2013).

These competing regimes present specific spillover effects which have been documented in Markov-switching rational expectations models (MSRE). Specifically the model framework presented by Farmer, Waggoner, and Zha (2011), focuses on the ability of their model to produce an indeterminate solution even though the required stability conditions were met. They recognized the importance of regime spillover and its ability to conquer traditional determinacy analysis. Furthermore, Ellison, Sarno, and Vilmunen (2007) attempt to explore this very idea by constructing a model which allows competing central banks to exploit the exchange rate regime by introducing Baysian learning. Though their treatment of exchange rates as exogenous processes, uninfluenced by monetary policy, rendered their work inadequate for future consideration. In a previous research, I explored new convergence criteria for adaptive learning models under the assumption of regime switching parameters. Unlike the research of Branch, Davig, and McGough which focuses on recalculated stability conditions founded in the work of Blanchard and Kahn (1980), my exploration follows the use of Mean-Square Stability to imply expectational stability in bounded rationality. I was able to construct tangible, succinct conditions to identify e-stability in MSV solutions. Empirically, I was able to associate the regime changes of the Federal Reserve as not e-stable, essentially showing that economic agents would not have learned the rational expectations outcomes during the currency interventions of the Plaza and Louvre accords.

The paper continues by developing the monetary model of state-dependent exchange rates under adaptive learning in the next section. Section 3 explore the monetary model using adaptive learning whilst comparing the theoretical results to the rational expectations solution. The last sections explores the exchange regime changes during the 1980s by outlining statistical occurrences during each regime change. Moreover, I present an analysis of convergence to e-stability under mean-square stability when a monetary shock is present. Concluding remarks are found in the final section.

MONETARY MODEL OF EXCHANGE RATES

Considering there are many iterations of the monetary model of exchange rates, I focus on a form with clearly defined monetary policy parameters. Similar to the monetary models of Frenkel (1976), Mussa (1976), and Kim (2008), Kumah (2008) supposes the connection of exchange rates through purchasing power parity (PPP) as well as uncovered interest parity (UIP). Let real money balances be defined as

$$m_t^d - p_t = \alpha y_t - \beta i_t + v_t \qquad (1)$$

where $(m_t^d - p_t)$, real money balances, is a log-linear function of income y_t, domestic interest rates i_t, and an unanticipated domestic money shock v_t. Thus, α is defined to be the income elasticity of money, and β is the interest semi-elasticity of money.

By assuming that PPP, $s_t = p_t - p_t^*$, and UIP, $E^*(\Delta s_{t+1}) = i_t - i_t *$, holds Eq. (1) becomes

$$m_t^d = (s_t + p_t^*) + \alpha y_t - \beta\big(i_t^* + E^*(\Delta s_{t+1})\big) + v_t \qquad (2)$$

where p_t^* is defined to be the log foreign price, s_t is the nominal exchange rate defined as the domestic price of foreign currency, and i_t^* is the foreign interest rate. The expectations operator, E^*, denotes time *t* expectations of the *t+1* change in nominal exchange rates under adaptive learning. The domestic money supply is defined to be the linear combination of domestic credit and foreign reserves. Assuming a multiplier of unity Eq. (3) represents the domestic money supply,

$$m_t = d_t + r_t \qquad (3).$$

In his analysis Kim disregards the monetary authority and the possibility of policy regime changes by failing to include an exchange rate policy parameter. Eq. (4) defines the relationship of the domestic monetary authority and their ability to intervene. The Federal Reserve in this instance would intervene in the market for foreign exchange in accordance to the policy rule

$$\Delta r_t = -\chi \Delta s_t \qquad (4)$$

where χ is the exchange rate sensitivity parameter. Policy would dictate selling foreign exchange as the exchange rate depreciates and purchasing as the exchange rate appreciates. In order to create the state contingent parameter, I employ a Markov process where z_t is only a three state Markov process where z_t evolves according to the transition matrix,

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$$

where $P = (p_{ij}) for\ i,j = 1,2$ with p_{ij} being the probability that $z_t = j$ given that $z_{t-1} = i$ and that

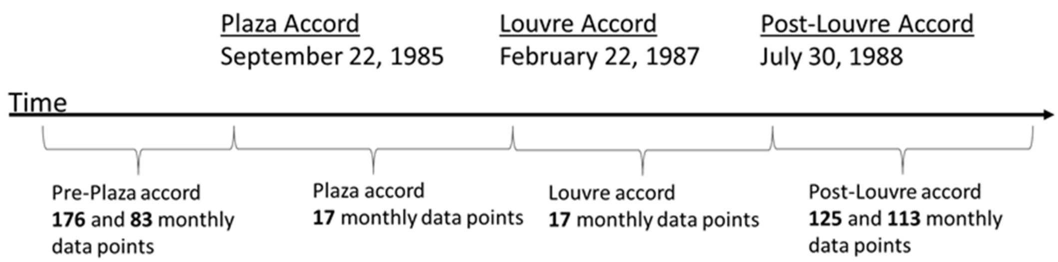
$$\sum_{i=1}^{3} p_{ji} = 1, for\ all\ j = 1,2,3.$$

The characteristics of the transition matrix are taken to be recurrent and aperiodic implying a unique stationary distribution. Eq. (4) now becomes,

$$\Delta r_t = -\chi(z_t)\Delta s_t. \qquad (5)$$

The monetary authority can choose between three different policy options: to actively appreciate, depreciate or to allow a free-floating exchange rate. Under this framework, as the exchange rate is depreciating, that is, the change in Δs_t is positive, the monetary authority can aid in the depreciation. This would require the degree of intervention to also be positive so that an interventionist central bank would be selling foreign reserve. On the other hand, if there appears to be pressure from appreciating exchange rates, that is, the change in Δs_t is negative, the monetary authority could aid in the appreciation by purchasing foreign reserve, requiring the degree of intervention to be negative. Lastly, if the degree of intervention is zero, then the monetary authority has chosen a free-floating exchange rate and there would be no foreign currency intervention. Empirically, during the Plaza and Louvre accords there was no pure free-floating exchange rates rendering the last state implausible during the time, but in the pre-plaza and post-Louvre accords this regime is probable. Figure 2 below shows a timeline of each currency market intervention.

Figure 2. Timeline of Currency Market Intervention



In order to rewrite the monetary model in a more appealing form, I first take the first difference of Eq. (2) and (3), so that money demand and supply now become,

$$\Delta m_t^d = (\Delta s_t + \Delta p_t^*) + \alpha \Delta y_t - \beta(\Delta i_t^* + E^*(\Delta s_{t+1}) + \Delta s_t) + \Delta v_t \qquad (6)$$

$$\Delta m_t = \Delta d_t + \Delta r_t \qquad (7).$$

To satisfy the e-stability conditions set forth by Evans and Honkapohja (2001) and the mean-square stability conditions created by Reed (2014), Eq. (6) and Eq. (7) can be rewritten as,

$$\Delta s_t = \Pi(z_t)(\beta \Delta i_t^* - \Delta p_t^* - \alpha \Delta y_t + \Delta d_t - \Delta v_t) + \Pi(z_t)\beta E^* \Delta s_{t+1} \qquad (8)$$

where $\Pi(z_t) = \frac{1}{(1+\beta+\chi(z_t))}$. The equilibrium exchange rate is now expressed as a function of the fundamentals and the monetary policy parameter. This framework is comparable to the approach created by Kim (2008), Evans and Chakraborty (2008), and Chakraborty (2009) where they express the log nominal exchange rate as a function of the fundamentals and forward spot exchange rate. Although these studies aim to explore the forward premium puzzle, it is important to note that they fail to account for changing policy decisions with state dependent parameter values. This analysis creates a bridge between Markov-switching regime changes and the forward premium puzzle and will be valuable for future work.

Furthermore, Eq. (8) follows the same intuitive understanding as classical models from Dornbusch (1976) and Branson and Henderson (1985). We can observe that foreign price and domestic output increases lead to appreciation of the domestic currency, as well as, positive domestic money supply shocks which increase the domestic interest rate. On the other hand, increases in foreign interest rates and increases in domestic credit lead to depreciating domestic currency. Expansionary monetary policy shocks will also lead to a depreciation in the exchange rate by lowing the domestic interest rate.

The policy parameter becomes an important component of the model dynamics and ultimately governs the asymptotic convergence to the rational expectations solution. The range of values for the exchange rate parameter can be anchored by free-floating intervention $\chi(s_t) = 0$, where policy is absent, and by holding exchange rates fixed, $\lim_{\chi(s_t)\to\pm\infty} \Delta e_t = 0$. Intuitively, intermediate policy can be explained then by having $0 < \chi(s_t) < \infty$. Kumah proposes that policy values which follow $-(1+\beta) < \chi(s_t) < 0$ represent a magnification of the changing exchange rate. On the other hand, when $\chi(s_t) < -(1+\beta)$ exchange rate policy is assumed to be aggressively leaning against the wind. To summarize the proposed policy regimes, assuming a multi-lateral relationship where countries coordinate with an agreed upon strategy and not unilaterally in competition with foreign central banks,

$$\chi(s_t) = \begin{cases} \chi_1 \; for \; z_t = 1, depreciation\; pressure \\ \chi_2 \; for \; z_t = 2, appreciation\; pressure \\ \chi_3 \; for \; z_t = 3, \qquad no\; intervention \end{cases}.$$

To simplify the analysis and to model the exchange rate climate during the 1980's, I assume that state $z_t = 3$ is not an active policy state but a plausible state nonetheless. An alternative assumption which I explore in further research would be for central banks to work in unison or to exploit the information spillovers created by both central banks to use the exchange rate for their advantage. This is the policy approach taken by Ellison, Sarno, and Vilmunen (2007).

The idea of regime spillovers creating the possibility of competing regimes highlights an important concept expressed by Farmer, Waggoner and Zha. That is, indeterminacy can develop in seemingly stable models due to the linearization of the non-linear model and the multiplicative interaction of the regimes. Linearizing Eq. (8) produces a system of equations which include the state dependent parameters and the transitional probabilities for the M-state Markov regime,

$$\Delta s_{t,1} = \Pi_1(\beta_1 \Delta i_t^* - \Delta p_t^* - \alpha_1 \Delta y_t + \Delta d_t - \Delta v_t) + \Pi_1 \beta_1 p_{11} E\Delta s_{t+1,1} + \Pi_1 \beta_1 p_{12} E\Delta s_{t+1,2} + \cdots + \Pi_1 \beta_1 p_{1m} E\Delta s_{t+1,m}$$

$$\vdots$$

$$\Delta s_{t,m} = \Pi_m(\beta_m \Delta i_t^* - \Delta p_t^* - \alpha_m \Delta y_t + \Delta d_t - \Delta v_t) + \Pi_m \beta_m p_{m1} E\Delta s_{t+1,m} + \Pi_\mathrm{m} \beta_m p_{m2} E\Delta s_{t+1,m} + \cdots + \Pi_m \beta_m p_{mm} E\Delta s_{t+1,m}$$

as expressed by Reed and Branch, Davig and McGough the system can be rewritten in a reduced form, forward looking, expectational difference equation:

$$\widehat{\Delta s_t} = \widehat{\Pi}\mathrm{H_t} + \Pi E_t \widehat{\Delta s}_{t+1} \qquad (9),$$

where $\mathrm{H_t}$ is the vector of fundamentals and $\Pi = (\oplus_{j=1}^{m} \beta_j)(P \otimes I_n)$ and the coefficient matrix Π is governed by $\oplus_{j=1}^{m} \beta_j = diag(\beta_1, \beta_2, \dots, \beta_m)$.

This result is important as it is the foundation for the mean-square stability conditions set forth in the following sections. Moreover, this form is ideal for purely forward looking expectational difference equations which use adaptive learning. In the next section, by using Eq. (9), I outline the basic features of adaptive learning and introduce the recursive least squares learning algorithm.

ADAPTIVE LEARNING, RATIONAL EXPECTATIONS, AND MSS

The framework for adaptive learning and the conditions which govern e-stability, i.e. the convergence of a learning outcome to rational expectations, have been extensively written upon with Evans and Honkapohja (2001) writing the most extensive survey of the literature. In a recent update to the literature Branch, Davig, and McGough look at the stability conditions which govern Markov switching regimes and subsequently produce results which are neither conclusive nor tractable. Yet, their foundation proved to be beneficial as it laid the work for using new techniques to ensure convergence to an e-stable outcome. Before the incorporation of adaptive learning and mean-square stability, the general framework for adaptive learning is briefly outlined in this section.

Bounded rationality can be introduced into the model by allowing market participants to only know the general framework of the model but do not know the policy parameter values. Agents will have to learn the values of the parameters by updating their estimates with new information as well incorporating their previous miscalculations. Agents begin by creating a perceived law of motions (PLM) where parameter values are not known but are estimated at time *t* using a learning algorithm,

$$Y_t = A + Br_t$$

$$and$$

$$E_t^* y_{t+1} = A_{t-1} + B_{t-1}\rho r_t.$$

The PLM has a unique feature built into its design as it is also the Minimum State Variable solution (MSV). McCallum (1983) first identified this solution design, which was then expanded by Evans and Honkaphoja in the context of adaptive learning and finally by Farmer, Waggoner, and Zha

(2011) in a Markov switching model. The idea of the MSV is that there exists no smaller set of linear dependent variables which provides a solution.[8]

Agents use their PLM and the information available to them, up to and including time *t-1,* in order to update the actual law of motions (ALM) when the parameters are realized in time *t+1*. Adaptive learning represents the ALM by substituting the PLM into the realized transition of the endogenous variables,

$$y_t = MA_{t-1} + (MB_{t-1}\rho + \eta)r_t.$$

Over time, the learning process continues in a similar format until either the parameter values diverge from the rational expectations outcome or converges. Convergence under adaptive learning is known as E-stability.[9]

Adaptive learning though requires more than positing values and then subsequently updating the information but must take into account the error of the initial guess. As Kim states, "a sensible strategy for market participants would be to estimate these parameters by linear least squares which will lead to a consistent estimate." Suppose Θ represents the actual parameter estimates realized by a market participant, then let $\widehat{\Theta}$ represents an agent's estimates of the parameters. Furthermore, agents will update their estimates using a learning method like recursive least squares (RLS) or constant gain learning (CG)[10]. Below I outline more rigorously the general method for developing RLS.

From Eq. (9) let the change in the exchange rate be a function of the fundamentals process H and an error term so that

$$\Delta s_t = \mathrm{H}'_{t-1}\Theta_{t-1} + \gamma_{t-1}\epsilon_t$$

[8] For further work and proof of MSV solutions see Evans and Honkapohja (2001) pg. 176 and Farmer, Waggoner, and Zha (2011)
[9] Evans and Honkapohja (2001) have an extensive summary of the process of learning.
[10] Constant gain learning requires agents to have an inherent updating parameter which orders the importance of the historical information.

After the realization of the parameters, agents run the regression of $\Delta s_t = \mathrm{H}'_{t-1}\Theta + \eta_t$ to obtain the ordinary least squares estimate of the coefficient, $\widehat{\Theta}$:

$$\widehat{\Theta}_t = \left(\sum_{i=1}^{t} \mathrm{H}_{i-1}\,\mathrm{H}'_{i-1}\right)^{-1} \sum_{i=1}^{t} \mathrm{H}_{i-1}\Delta s_i,$$

where we further define that

$$R_t = \frac{1}{t}\sum_{i=1}^{t} \mathrm{H}_{i-1}\,\mathrm{H}'_{i-1}$$

so the recursive expression of the OLS estimator can be written as Eq. (10) and (11),

$$R_t = R_{t-1} + \tfrac{1}{t}(\mathrm{H}_{t-1}\mathrm{H}'_{t-1} - R_{t-1}), \qquad (10)$$

$$\widehat{\Theta}_t = \widehat{\Theta}_{t-1} + \tfrac{1}{t}R_T^{-1}\mathrm{H}_{t-1}(\Delta s_t - \mathrm{H}'_{t-1}\Theta_{t-1}). \qquad (11)$$

The RLS formulation allows agents to use their forecast errors to proportionally adjust their estimates moving forward, but ultimately become less systematic as market participants learn the rational expectations equilibrium. Kim argues and I agree that this methodology appears to coincide with strong empirical evidence that economic agents are able to learn parameter estimates given a sufficient period of time even when agents initial parameter values of the fundamental process are very different from the rational expectations equilibrium. Thus RLS and learning depend heavily on the ability of the market agents to estimate close the RE parameter value. Since convergence is not always guaranteed the next section explores the conditions for e-stability.

MEAN SQUARE STABILITY AND E-STABILITY

Now that agents are estimating parameter values using an adaptive learning algorithm, the conditions of e-stability need to be briefly outlined. The coefficient matrix in Eq. (9), ultimately governs the stability of the model. In a multivariate, constant parameter model, the eigenvalues of the matrix must be less than one in modulus in order for e-stability to occur. These conditions, developed by Blanchard and Kahn (1980), have been extensively researched in linear models but under the state-dependent parameters additional conditions must be met to insure convergence. Taken from Markov switching rational expectations (MSRE) literature Mean-square stability (MSS) provides the foundation for the adaptive learning solution to converge to the solution guided by rational expectations.

Considering that MSRE models are governed by non-linear changes in the parameter, they exhibit properties which can contribute to the indeterminacy of the equilibrium outcome. The first being that the errors are serially correlated over time and thus should be reflected in the second moment matrix of the error term. Furthermore, solutions can be influenced by “sunspot” equilibria thus further contributing to indeterminacy. As a result, forward looking regime switching models can be represented by a fundamental and non-fundamental component. Lubik and Schorfheide (2004) provide the necessary framework to represent a forward looking model into the fundamental and non-fundamental components. If the Blanchard and Kahn conditions are met, these sunspot solutions tend to be stationary but regime switching models can still violate these conditions because of the spillover effects the regimes create. These spillover effects occur because of the a priori assumption that shocks are small and bounded in the neighborhood of the perfect foresight linear approximation. But the inherent nature of regime switching parameters violates this assumption. As Farmer et al. indicate, shocks in regime switching models are considered large when compared to perfect foresight model shocks since they move the state variables into a different region of the state space.

Because of its ability to include the first and second moments into the stability conditions, mean-square stability is standard usage in MSRE models and is defined to be,

Definition 1 *An $n\ x\ 1$ stochastic process y_t is mean square stable (MSS) if there exists an n x 1 vector $\bar{y}$ and an n x n matrix Q such that* $\lim_{t\to\infty}(E[y_t]-\bar{y}) = 0_{n\,x\,1}$ *and* $\lim_{t\to\infty}(E[y_t y_t']-Q) = 0_{n\,x\,n.}$

Definition 2 *An n-dimensional process y_t is bounded if there exists a real number N such that* $\|y_t\| < N, for\ all\ t.$

where $\|*\|$ is a well-defined norm.

This condition is stronger than traditional stability conditions which only rely on bounded mean analysis rather than imply the existence of the second moment matrix. It can be noted that there are alternatives to MSS such as covariance stationarity or asymptotic covariance stationarity. Both are slightly weaker conditions of MSS considering asymptotic covariance stationarity implies MSS but not conversely.

From this definition, the coefficient matrix governing Eq. (8) must be rewritten in the fundamental form proposed by Lubik and Schorfheide. That is, the

The stochastic process, Δs_{t+1}, is mean-square stable if and only if the spectral radius of the linearized coefficient $\overline{\Psi}_{G\otimes G}$ is less than one. That is $r_\sigma(\overline{\Psi}_{G\otimes G}) < 1$.

Adapting Theorem 1 to the MSRE models requires the fundamental solution have the same form as stated above, again assuming the vector of shock terms is already mean-square stable. That is $x_t = \Omega(s_t)x_{t-1} + \Gamma(s_t)z_t$ is mean square stable if and only if

$$r_\sigma(\overline{\Psi}_{\Omega\otimes\Omega}) < 1,$$

As for the non-fundamental component $w_t = \Lambda(s_{t-1}, s_t)w_{t-1} + V(s_t)V(s_t)'\eta_t$, w_t is mean-square stable if and only if

$$r_\sigma(\overline{\Psi}_{\Lambda\otimes\Lambda}) < 1.$$

Again, we assume the sunspot error term is white noise. Thus we get the following theorem.

Theorem 2 *The stochastic process, with both fundamental and non-fundamental components is found to be uniquely determinate under mean-squared stability if and only if* $r_\sigma(\overline{\Psi}_{\Omega\otimes\Omega}) < 1$, *and* $r_\sigma(\overline{\Psi}_{\Lambda\otimes\Lambda}) < 1$.

Proof. *See Definition 1.*

In previous research I present the link between MSS and e-stability, Reed (2014) by using the MSV solution where I prove the following preposition[11],

Proposition 1 *A unique MSV-solution which satisfies the conditions for mean-square stability is also E-Stable.*

This connection will be the basis for the empirical work in the following sections. Since agents will be assumed to follow a bounded sense of rationality, Proposition 1 will be required to analyze the effects of a monetary policy shock.

The remainder of the paper extends the foundations of Kim's simulation work by including the realization that state dependent parameter should produce a result which is expectationaly unstable. This result follows directly from the theoretical implications from the previous section and the research created in Reed (2014). The next section begins by exploring the process of simulating the exchange rate and the Federal Reserve's exchange rate policy parameter.

ESTIMATION AND SIMULATION

Through this section I outline the estimation of and subsequent simulation of the Federal Reserve's policy parameter and exchange rate with Japan. For this analysis I assume that the two countries are working bi-laterally to adjust the exchange rate. Although the Plaza and Louvre accords were the joint effort of England, France, USA, Canada, Japan, and Italy, the assumption of bi-lateral coordination is purely for the sake of simplicity. Furthermore, I examine how the simulated data has the capability of both qualitatively and quantitatively representing the

[11] For the full proof, please refer to Reed (2014)

YEN/USD exchange rate in the absence of a regime change but fails to replicate the data when introduced to different policy regimes.

Some descriptive statistics for the monthly exchange rate of the US dollar (USD), UK pound (LBS), German deutschemark (DEM), Japanese Yen (YEN), and the Swiss franc (FRA) are shown in the table below[12].

The data are monthly observations from 1978:10 to 1998:12 and were obtained from the St. Louis Federal Reserve FRED database. The exchange rates represent monthly averages of the foreign currency to one U.S. dollar. The foreign reserves for each central bank represent the monthly unilateral foreign currency holdings in domestic denominations.

Table 1. Exchange Rate Variance Ratios

	YEN	DEM	SWISS	POUND
mean	5.059	0.644	0.485	-0.512
std. dev	0.328	0.213	0.208	0.153
VR(1) n=223	1	1	1	1
VR(8) n=223	1.844	1.94	1.774	1.793
VR(16) n=215	2.304	2.578	2.156	1.931
VR(20) n=211	2.156	2.526	2.055	1.78

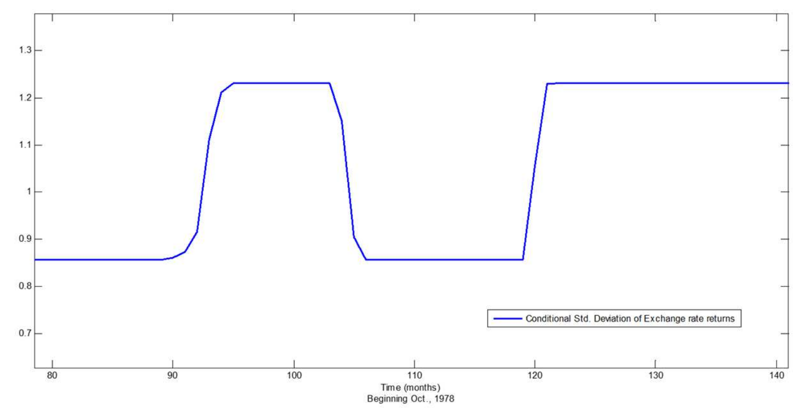
note: The data are monthly observations from 1978:10 to 1998:12 and were obtained from the St. Louis Federal Reserve FRED database. The variance ratio statistic, VR(*k*), is the variance of the *k*-monthly change divded by *k* times the variance of the one-month change.

Table 1 identifies the variance ratio statistic, VR(*k*) for the exchange rate of each national currency to the USD. This statistic is created by finding the ratio of the *k*-month change to the *k* times the ratio of the one-month change. Using the method pioneered by Lo and MacKinlay (1988, 1999) each exchange rate series exhibits a variance ratio greater than unity and appears

[12] Research indicates that exchange rates are not affected by seasonality thus all data has not been seasonally adjusted.

to increase as the time horizon increases and remain above unity. This behavior is indicative of serial correlation in the exchange rate over time and mean diverting performance. If the variance ratio statistic exhibited decreasing values and values below unity, as the time horizon increases, one could assume mean reversion. Kim finds that over a similar time frame, the quarterly exchange rate returns for the UK, Germany and Japan experience mean reverting behavior in the long-run, an obvious departure from my findings. Furthermore, Figure 3 shows the YEN/USD changes in conditional volatility during the regime changes with higher conditional volatility following the plaza accord regime change. The Louvre accord shows a decrease in the conditional volatility until the policy expires where there is a marked increase in volatility, which remains for the length of the sample period. Moreover, the adjustment of the conditional variance happens slowly over 7 months during the first regime change where the variance during the Louvre accord and post-louvre accord adjust more quickly, within a few months.

Figure 3. Conditional Std. Deviation of Exchange Rate Returns (YEN/USD)



These findings begin to provide evidence for the hypothesis that regime changes during the 1980s created excessive volatility. Furthermore the persistent long-run deviations from the mean (periods greater than 17 periods apart) coupled with the statistically different regime variances point to the possibility that agents are learning the regime changes slowly if at all.

In order to make direct comparisons between studies, this paper assumes that the interest semi-elasticity of money to be one and that the central bank's regime probability matrix needs to be estimated. In previous literature from Kim, the transition probability elements are provided ad hoc instead of the elements being estimated using MSRE techniques.

Further parameterization of the model also requires the estimation of the monetary policy parameter, χ. Robust, OLS regression estimates for the Federal Reserve holdings of YEN during each regime yielded results similar to the Markov-switching maximum likelihood estimation. Table 2 reports the results.

Table 2. The Federal Reserve's Holdings of YEN
Robust OLS estimates

	Pre-Plaza Accord	Plaza Accord	Louvre Accord	Post-Louvre Accord
χ	2.837*	-1.922**	7.574*	1.18
	(1.367)	(0.614)	(3.421)	(0.769)
n	83	17	17	113
R-Squared	0.05	0.38	0.23	0.02
ML Estimate	4.678	-22.615	5.661	5.661
	(0.00)	(0.00)	(0.00)	(0.00)

Note: Regressions were run with a suppressed constant term. * refers to estimates being significant at the 95% confidence level, while ** refers to significance at the 99% confidence level. For the MLE, three states were shown to be the maximum number of reasonable states, with state 1 representing the floating exchange, state 2 representing the Plaza accord, and state 3 the Louvre accord.

The robust OLS estimates provide an intuitive starting point for interpreting the Federal Reserve's exchange rate sensitivity parameter. The low R-Squared values for both the pre-Plaza accord and the post-Louvre accord point to the free floating nature of the exchange rate once policy intervention subsided. Moreover, the directions of the four coefficients match up the theoretical interpretation of the model. The magnitudes, representing the elasticity, shows a very elastic response to changes in exchange rates except for the post-Louvre accord period. Again, this points to an absence of active policy and move back to a free-floating exchange rate.

The Markov-switching maximum likelihood estimation corroborates the story being formed by the OLS estimates. The directions for each state again correspond with the theoretical implications of the model. Furthermore, the values of the pre-Louvre accord parameter are quantitatively similar between estimation methods. Moreover, the transition to the deprecation state indicates a more aggressive ML estimation of depreciation of the exchange rate than the OLS provided. The magnitude of the MLE parameter estimate is also approximately an order higher than that of the OLS. These similarities between estimates highlights one of the key findings of this research.

REGIME CHANGE ESTIMATES

Again assuming that the number of states, $z_t = 3$, so that the non-linear system reduced to the linear form of,

$$\Delta s_{t,1} = \Pi_1(\beta_1 \Delta i_t^* - \Delta p_t^* - \alpha_1 \Delta y_t + \Delta d_t - \Delta v_t) + \Pi_1 \beta_1 p_{11} E\Delta s_{t+1,1} + \Pi_1 \beta_1 p_{12} E\Delta s_{t+1,2} + \Pi_1 \beta_1 p_{13} E\Delta s_{t+1,3}$$

$$\Delta s_{t,2} = \Pi_2(\beta_2 \Delta i_t^* - \Delta p_t^* - \alpha_2 \Delta y_t + \Delta d_t - \Delta v_t) + \Pi_2 \beta_2 p_{21} E\Delta s_{t+1,2} + \Pi_2 \beta_2 p_{22} E\Delta s_{t+1,2} + \Pi_2 \beta_2 p_{23} E\Delta s_{t+1,2}$$

$$\Delta s_{t,3} = \Pi_3(\beta_3 \Delta i_t^* - \Delta p_t^* - \alpha_3 \Delta y_t + \Delta d_t - \Delta v_t) + \Pi_3 \beta_3 p_{31} E\Delta s_{t+1,1} + \Pi_3 \beta_3 p_{32} E\Delta s_{t+1,3} + \Pi_3 \beta_3 p_{33} E\Delta s_{t+1,3}$$

so that the solution vector is defined to be, $\widehat{\Delta s_t} = [\, \Delta s_{t,1}, \Delta s_{t,2}\,, \Delta s_{t,3}\,]$, the OLS estimates of the regime changes can be used to estimate the Markov switching regimes. Thus, z_1 can be thought of as the floating exchange rate, z_2 is the Plaza accord, and z_3 would be Louvre accord.

Figure 4. Estimated Regime Transitions for YEN/USD Exchange Rate

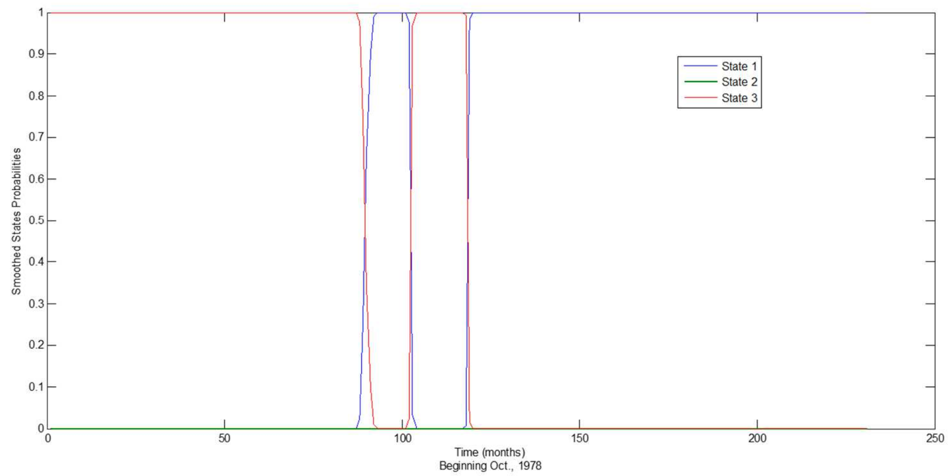


Figure 4 approximates the smoothed state probability estimates from the MLE parameter estimates. As expected the Markov-switching regime changes align perfectly with each hypothesized regime change during the Louvre and Plaza accords. This finding is again an encouraging outcome as it provides additional support for the validity of the MLE and OLS estimates. Moreover, it becomes apparent that the pre-Plaza accord horizon was marked with appreciating exchange rates, something that was expressed as the pre-plaza accord exchange rate bubble. The underlying transition moves through Plaza accord but since the actual state was not absorbing, then moves on to the Louvre accord and finally ends with the free floating exchange rate. Table 3 below displays the estimated transition matrix. What is interesting about the transition probabilities is that moving from state 2, the Plaza accord, to state 3, the free floating exchange rate would be absorbing.

Table 3. Estimated Regime Transition Probabilities

$$P = \begin{matrix} & 1,_{t+1} & 2,_{t+1} & 3,_{t+1} \\ 1,t \\ 2,t \\ 3,t \end{matrix} \begin{bmatrix} 0.99 & 0 & 0.01 \\ 0 & 0 & 1 \\ 0.02 & 0 & 0.98 \end{bmatrix}$$

MEAN-SQUARE STABILITY

In order to include MSS into the model a researcher can choose one of two ways to build state dependent parameters into the analysis. The first would be to run static learning parameter estimates using the known regime demarcations and then observe the parameter evolution when a regime change occurs. This would effectively simulate the exchange rate dynamics in three separate and distinct observations. Kim's research relies upon this framework considering his e-stability conditions fail to allow for the spillover effects that each regime change creates. The second is to run a dynamic learning environment, which by linearizing the model builds into the framework separate regime exchange rate variables. Again, this algorithm is preferred as it allows for the regime spillover effects which are present throughout the estimation.

This apparent spillover can be seen in the failure of the Federal Reserve's regime changes to meet the MSS conditions. The results from Reed (2014) are displayed in Table 4 and show that the policy changes that the Federal Reserve took during the Plaza and Louvre accords would not have resulted in learning the rational expectations solution.

Condition for stability: $r_\sigma(\bar{\Psi}_{\Omega\otimes\Omega}) < 1$, *and* $r_\sigma(\bar{\Psi}_{\Lambda\otimes\Lambda}) < 1$

Table 4. Regime changes for the Federal Reserve

	Pre-Plaza to Plaza Accord	Plaza to Louvre Accord	Louvre to Post-Louvre Accord
$r_\sigma(\bar{\Psi}_{\Omega\otimes\Omega})$	*0*	*0*	*0*
$r_\sigma(\bar{\Psi}_{\Lambda\otimes\Lambda})$	*147.92**	*131.49**	*0.09*

Note: * refers to non-stable monetary policy during regime switches

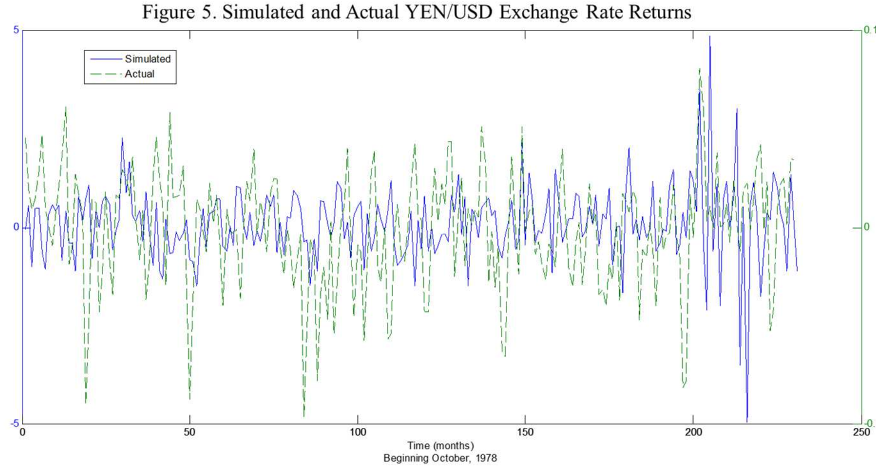
It is important to note that to move forward with simulating the learning environment one must reconcile the timing of the agents' information set. Generally when using timing mechanisms an additional level of assumption may be needed, that is agents with information sets up to an including *t-1* are required to forecast non-observable variables. This could be accomplished with agents using the Kalman filter to forecast time *t* variables in time *t-1*, but would require some inherent knowledge as to which non-observable variables should be included in the forecast. This added assumption deviates from the spirit of adaptive learning and what this paper is trying to accomplish. Thus, to eliminate this assumption, VAR style learning can be used to recursively express time *t* variables.

Furthermore, I rely on arbitrarily fixed initial beliefs, as they are the simplest way of initializing beliefs and intuitively seem to reflect an agents information set. This is done through a subsample regression and then compared to a standardized identity variance-covariance matrix. This assumption is important as it doesn't require that agents begin with rational expectations beliefs. RE consistent beliefs would imply that the model was solved originally using rational expectations parameters and the covariance matrix. This is a clear violation of bounded rationality and as such, this analysis restricts initial believes to be arbitrarily fixed.

ADAPTIVE LEARNING

Before the inclusion of adaptive learning into the process, I first analyzed how well the MSRE model simulated the real data. In Figure 5 below, I find that the MSRE process, parameterized per the findings above drastically overshoots the changing YEN/USD exchange rate but seems to capture the qualitative structure of the actual series.

Figure 5. Simulated and Actual YEN/USD Exchange Rate Returns

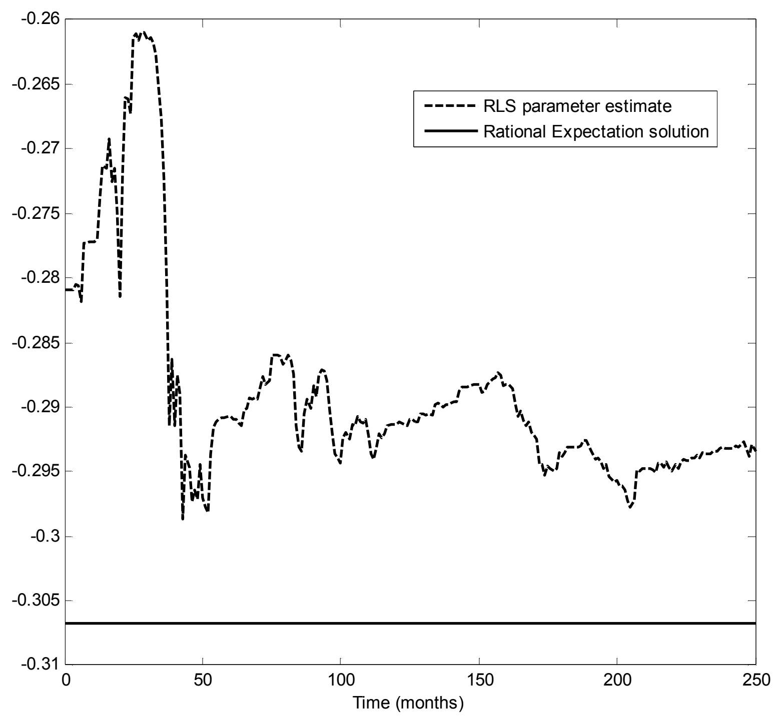


It will be shown that by including adaptive learning into the simulation, the actual data and the simulated data are more consistent. This points toward the importance of adaptive learning in the model.

In a recent paper regarding the use of adaptive learning in forward looking models, Carceles-Poveda and Giannitsarou (2007) generate code which helps users explore the different adaptive learning algorithms: recursive least squares (RLS), stochastic gradient (SG), and constant gain (CG). By modifying the language I am able to institute regime-switching parameters and found that the agents were not able to learn the Federal Reserve's policy parameter. Therefore the learning solutions are not e-stable. This is an important outcome since it corroborates the outcome found previously when using MSS conditions. The remainder of this section highlights key iterations of the learning solution using a various learning algorithms.

Solving for the rational expectations solution indicated that the parameter value to be learned is -0.3067. Moreover, considering that the parameter estimates do not yield a MSS solution, the learned solution should not converge to the RE outcome over the course of the estimation. I begin with the RLS solution by assuming that the shock vector is random with a variance of one. I simulate the data over 250 periods; similar to the length of the exchange rate series 231. The initial starting point for the RLS solution is assumed to be at the average parameter estimate. Figure 6 below shows the RLS parameter estimate.

Figure 6. Federal Reserve Ex. Rate Sensitivity RLS and RE Parameter Estimates



The RE solution corresponds to the Federal Reserve's exchange rate sensitivity parameter being estimated at -5.27 while the RLS estimation fluctuates between -5.84 and -5.39. It appears that the estimates overshoot the RE solution at the beginning of the simulation where they never quite recover and fail to converge. Moreover, it appears that estimates throughout the simulation appear to overestimate the parameter value. This result coincides with the idea that agents would not have been able to learn the Federal Reserve's policy parameter during the regime switching. Furthermore, additional simulations were conducted using a variety of learning combinations with each estimation exhibiting similar results.

Figure 7 outlines the stochastic gradient simulation. The SG estimate again initially overshoots the RE solution and begins to converge while oscillating around the RE solution. This continues until just after the 100th month where then the estimate diverges and begins to fluctuate with higher volatility. This divergence appears to correspond with the estimated and hypothesized regime switches. Both the RLS-CG[13], in Figure 8, and RLS-SG parameter estimates show severe divergence from the RE solution and also appear to both exhibit some oscillation around the RE solution. Regardless, the RLS-SG parameter estimate displays a very high degree of variation which seems to compound as the simulation moves through the 150th period and intensifies at the end of 250 time periods. Moreover, the RLS-CG estimate begins to converge after an initial divergence but seems to again experience high degrees of variation when the regime switches and finally explodes at the end of the estimation.

[13] Its important to note that changes in the gain parameter exhibited no real difference in the learning outcome. As a result I parameterize the algorithm with the value of 0.3, which is a widely accepted value in the learning literature.

Figure 7. Federal Reserve Ex. Rate Sensitivity RLS-SG and RE Parameter Estimates

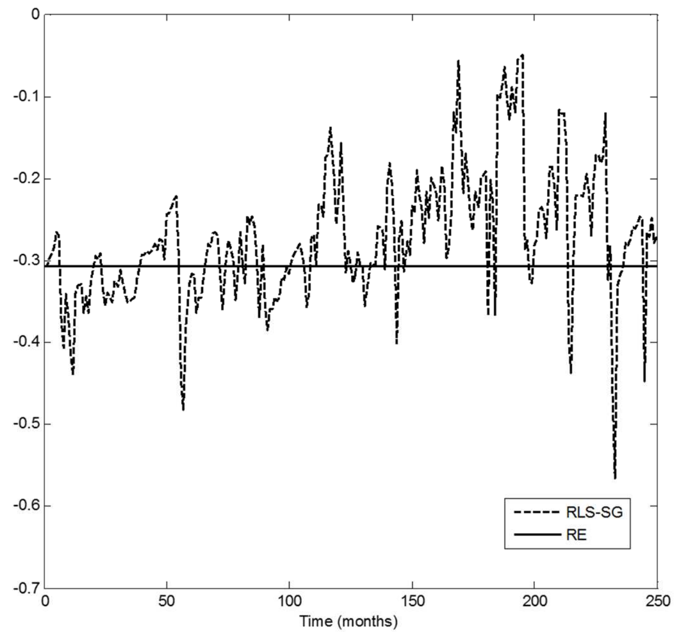


Figure 8. Federal Reserve Ex. Rate Sensitivity RLS-CG and RE Parameter Estimates

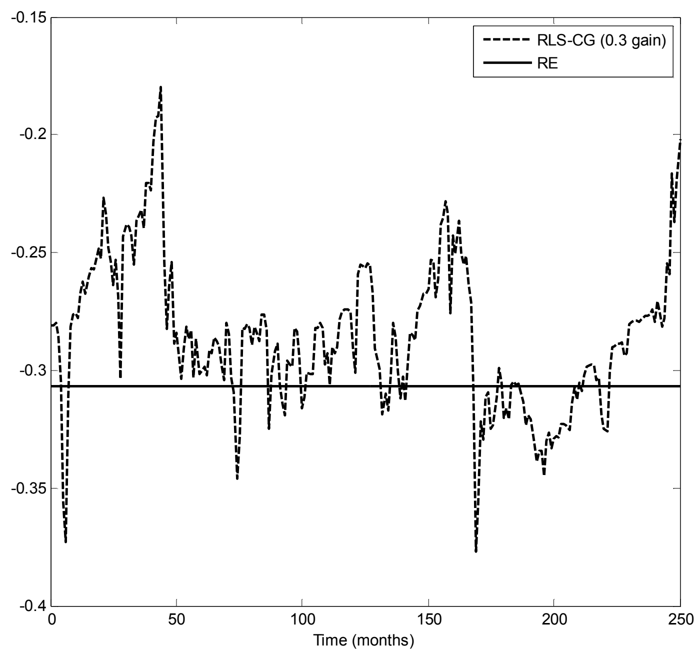
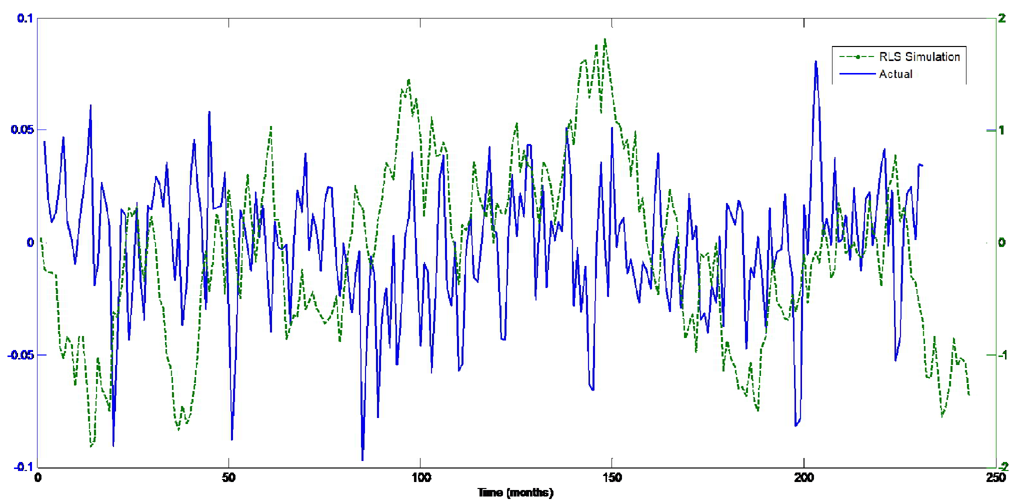


Figure 9 describes the simulated exchange rate returns series under adaptive learning. The simulated returns are conditioned on starting in an appreciating exchange rate regime. This is so the exchange rate climate in the simulation coincides with the actual experiences of the exchange rate. Overall the simulated returns appear to overemphasize the depreciation of the exchange rate throughout the horizon but show some indication of qualitative and quantitative equivalence. At the beginning of the simulation, AL appears to recreate the volatility and the appreciation of the actual YEN/USD returns but then fails to closely approximate the returns after the initial regime change occurs. The contributions of adaptive learning and state dependent parameters again suggest that the excess volatility comes from these underlying model assumptions. For the remainder of the time horizon the actual exchange rate series looks qualitatively to the simulation. Again, this evidence corroborates the idea that Markov-switching regime changes which feature adaptive learning are an ideal model in the presence of bounded rationality.

Figure 9. Actual and Simulated Federal Reserve Ex. Rate Returns



CONCLUDING REMARKS

For this analysis, I employ a standard monetary model of exchange rates and attach to it the features of adaptive learning and state-dependent parameters. By governing the dependency, using a Markov-switching process, I am able to use the foundation of mean-square stability in order to analyze the adaptive learning estimates. What I find is that economic agents during the Plaza and Louvre accords would not have been able to learn the Federal Reserve's parameter decisions and thus create a climate of excess volatility in YEN/USD exchange rate returns. This unique research is predicated on the idea that agents must be bounded in their rationality, a hypothesis that macroeconomics has been slow to adopt. This boundedness provides an empirical hypothesis, different from modern macroeconomics, which believes that agents truly incorporate misspecification, sub-optimal decision making, and systematic errors into their decision processes. Adaptive learning feels like an intuitive alternative to rational expectations and appears to produce tangible results.

In order to link rational expectation to that of adaptive learning through e-stability, this paper uses the necessary conditions of mean-square stability. Earlier work has shown that MSS provides a stronger, tractable link to expectational stability, something Markov-switching adaptive learning models had been previously missing.

The monetary model of exchange rates allows for a deeper analysis regarding the effects adaptive learning and Markov-switching have on economic agents. Given the assumptions which accompany these features, this paper finds greater volatility and slower convergence under adaptive learning than found traditionally with rational expectations. Moreover, I find that agents overestimate the depreciating effects of the Federal Reserve's exchange rate policy parameter. When compared to OLS estimates of approximately -2, agents using adaptive learning estimated the elasticity to be -5, rendering them much more sensitive to the depreciating exchange rate. Intuitively, this estimation may be a result of the drastic effects from the Plaza and an important insight into how agents estimated future returns.

Although not a novel finding, this paper combines the use of adaptive learning and state-dependent parameters in a way which is unique to the literature. Moreover, I find that adaptive learning without a change in regimes does fairly well at modeling exchange rate returns but again overestimates returns once the initial regime changes. Furthermore, recursive least squares proves to come closest in learning the rational expectation solution. Other algorithms like RLS-CG and RLS-SG fail to produce any semblance of a RE solution with mostly divergent outcomes. Again the intuition rests in the fact that once the regime has switched, agents are very slow to react, often overshooting the actual value of the parameter.

Future policy decision would benefit from this research as it shows realistically how economic agents absorb and use information. This would be valuable to a policy maker trying to execute a specific economic policy as it could help to produce a more reliable outline as to how agents react to forecasting future macroeconomic variables.

CHAPTER 3 "WHAT CAN MEAN-SQUARE STABILITY, ADAPTIVE LEARNING, AND REGIME SWITCHING TELL US ABOUT THE FORWARD-PREMIUM PUZZLE"

INTRODUCTION

Since its initial treatment, the forward premium puzzle has been a longstanding paradox within macroeconomic and finance research. The puzzle corresponds to the fact that empirically the forward exchange rate is a poor predictor of the expected depreciation in the spot exchange rate. Under rational expectations (RE), the OLS estimate is consistently underestimated in magnitude and often is estimated to be negative. As stated by Chakraborty and Evans (2008), the forward-premium puzzle is surrounded by additional stylized empirical results. The goodness-of-fit measurement, R^2, of the forward-premium regression is generally low, while the estimate of the forward premium is positively correlated. Chakraborty and Evans contend that econometric learning plays a large part in the abolishment of these poor results. They find that learning not only generates the forward-premium puzzle but also creates the stylized empirical results which surround most attempts.

Additional motivations for explaining the forward premium puzzle can be split into two camps: investor risk aversion and non-rational expectations. Risk aversion in the foreign exchange market incorporates the idea that investors need a risk premium to invest in a volatile asset to hedge against the risk. Although this offers an intuitive explanation, empirically this result fails to explain the puzzle. Evans and Chakraborty turn to the second approach to motivate their paper, non-rational expectations. Non-rational traders could potentially distort asset prices away from fundamental values thus creating the low predicting power seen in ordinary least squares regressions. First identified as a potential explanation by De Jong et al. (1990), Mark and Wu (1998) show that noisy traders under certain assumptions mimic the empirical data.

Like Chakraborty and Evans, Kim (2009) insists that adaptive learning (AL) contributes to the volatility experienced in the forward premium. In traditional analysis of exchange rate movements, RE fails to account for the volatility in economic fundamentals. Literature suggests that deviations from rational expectations or the underlying assumptions of the model are required for explaining the large volatility[14]. Moreover, what seems most puzzling is why RE fails to predict exchange rate fundamentals over a short period while doing very well at predicting fundamentals over long time horizons.

In recent years, researchers have attacked the assumption of rational expectations calling into question the ability of an economic agent to perfectly estimate parameter values econometrically. The idea of bounded rationality has grown into a substitutable assumption for RE. In Kim's paper, he suggests that adaptive learning is superior to RE for modeling the forward premium puzzle. I build on his research by including a more rigorous approach to adaptive learning by including the possibility of state dependent parameter values. Furthermore, I introduce mean-square stability as a necessary condition for assessing the economic agents' learning process. This last addition allows for the very realistic possibility of regime switches within monetary policy.

This paper is motivated by the latter idea, that non-rational agents are moving the fundamental values away from rational expectations in the foreign exchange markets. This

[14] See Meese and Singletom (1986), West (1987), MacDonald and Taylor (1994) and Shiller (1987)

influence of bounded rationality can be expressed as the downward bias of the OLS regression on the forward premium. This paper aims to replicate the results found in previous non-rational expectations literature as well as provide additional motivation for the observed bias. Furthermore, the literature suggests that the number of empirical observations have a strong effect on the magnitude of the downward bias on the estimated parameters. Intuitively, this influence is ingrained in the tradition of non-rational behavior. Economic agents participating in the foreign exchange market rely on a specific information set in order to forecast future spot exchange rates. As a regime change occurs, the observations cultivated to make forecasts suddenly become less reliable. Having an adequate amount of information would help agents to learn the rational expectations solution.

To test these assumptions empirically, this paper looks at recent interventions in the foreign exchange market for possible regime changes. Over the course of three decades, there have been isolated incidents of marked currency market intervention. Although the reality is that most exchange rate series are susceptible to currency manipulation. Most recently, this past January the Switzerland national bank made an unexpected decision to unpeg the Swiss Franc from the Euro. What was once considered a very stable monetary environment quickly ushered in a period of panic and uncertainty. Over the course of the announcement, the Swiss stock market collapsed and hedge funds recorded big losses, as the exchange rate appreciated by approximately 30%. So why did this happen? During the period of pegged exchange rate, investors sought out cheaper Francs, ultimately appreciating the currency and putting Switzerland's export heavy economy in danger of faltering.[15] Although this event contains a specific demarcation in regimes it offers little in the way of continuous regime uncertainty. This paper uses this empirical observation to understand the influence regime changes have on bounded rationality.

[15] On 1/15/15 the Switzerland National Bank (SNB) unpegged the Swiss Franc to the Euro Zone Euro. The Economist provided the information and I refer readers to their coverage for more topical analysis.

Furthermore, announced currency market interventions were observed during a tumultuous time during the 1980's in an effort to appreciate the German Deutsche Mark (DEM) and the Japanese Yen (YEN) in response to the U.S. dollar (USD) appreciating aggressively. Known as the Plaza and Louvre accords, multi-lateral currency intervention achieved the significant depreciation of the USD against the DEM and YEN. In fact it worked so well, the Louvre accord was established to stabilize the depreciation. The effects of which were felt long after the final intervention in 1987-88.

In this paper I find that the inclusion of state-dependent parameters represents the exchange rate fairly accurately. I am able to simulate the estimated coefficients found empirically for the U.S., U.K., Japan, and Switzerland. Moreover, certain parameterizations achieve the theoretical value of the premium coefficient from the projection of the exchange rate onto the forward premium. These inclusions appear to better characterize exchange rate behavior over the last few decades. That is, exchange rate fundamentals have likely been susceptible to changes in underlying parameter values through various channels. Oil price shocks, regime changes in monetary policy and monetary realignment all have impactful effects on exchange rate processes.

CURRENT LITERATURE

In their recent paper, Evans and Chakarborty discuss the merits of adaptive learning and how this deviation from rational expectations can help explain empirically slow convergence of the forward premium to the rational expectations solution. They believe that agents rely on learning, in the econometric sense, to form expectations regarding the forward exchange rate. It is because of this deviation from rational expectations the authors have been able to show that adaptive learning reproduces key empirical results in the data that is often attributed to irrationality in the exchange markets. Evans and Chakraborty assume that agents use a rolling understanding of parameter values to obtain expectations close to rational, assuming agents anticipate structural changes. This paper is partially motivated by their exclusion of state-dependent parameters in an attempt to explain the downward bias in estimated forward-premium regression coefficient that Fama (1984) first identified. They suggest that the foundation they constructed within the monetary exchange-rate model can be beneficial for more elaborate models using perpetual learning. Although, they suggest that extensions of their work should come from assumptions regarding risk aversion, incomplete price adjustment, heterogeneous expectations and incomplete information processing, I employ a state-dependent parameter model and explore the convergence of expectations under mean-square stability.

Using the canonical monetary exchange-rate model, Kim furthers the literature by briefly including state-dependent parameters under adaptive learning. Kim argues against the rational expectations paradigm by noticing that the movements in the volatility of exchange rate data could not be "justified by movements in economic fundamentals." These assumptions though require conditions for stability which appropriately account for the multiplicative nature of regime switching. Kim fails to provide the correct stability analysis. Also, the method for employing the regime change is limited to simulating the learning environment with a strict change in parameter values instead of including the transition probabilities in the model. A recent paper presented by Branch, Davig, and McGough (2013) provides convergence and stability conditions for Markov switching adaptive learning models, but shown by Reed (2014) these conditions are not rigorous nor tractable under these circumstances.

Moreover, empirically, rational expectations has proven time and time again to fail in its ability to accurately predict exchange rate dynamics over the short run. Kim continues the argument found in Reed (2014), as well as, Evans and Honkiphoja (2001) that rational expectations appears to be too strong an assumption for exchange rate fundamentals. In practice, economists must estimate econometrically parameter values; so why do we assume agents within the model are able to do the same? Adaptive learning allows for the possibility of expectations to be near rational. Market participants should assumed to have some limited knowledge about the true economic structure and a plausible view of the parameter value which needs to be estimated.

Kim, motivated by adaptive learning, again employs the monetary model that Evans and Chakraborty use in order to show that the convergence to rational expectations is slow and that adaptive learning is a plausible alternative to rational expectations. The results of Kim's analysis shows that adaptive learning provides insights into three main tenants. The first is that adaptive learning appears to outperform rational expectations when predicting exchange rate returns over long horizons. Secondly, Kim shows that adaptive learning can generate empirically similar exchange rate volatility in excess of fundamentals volatility. Finally, it is shown that adaptive learning is able to produce persistent deviations of the exchange rate from the fundamentals. Intuitively, these last two outcomes reveal that agents slowly absorb new information and that adaptive learning creates the possibility of slow convergence to rational expectations. This paper aims to replicate the results found by Kim, Evans and Chakraborty, and Chakraborty in that the convergence of the exchange rate estimated by adaptive learning to rational expectations is not necessary guaranteed under mean-square stability and state-dependent parameters. Furthermore, adaptive learning under Markov switching parameters provides an excellent explanation of the forward premium puzzle.

The remainder of the paper is organized as follows. The next section introduces the monetary model used for exchange rate analysis which includes adaptive learning dynamics along with state-dependent parameters. Section 3 reviews the stability conditions for e-stability and explores more stylized facts regarding exchange rate. Moreover, I simulate the economy under a set of parameterized values in order to compare the results from a Markov-switching Adaptive Learning (MSAL) model to empirical observations. The final section contains concluding remarks and observations.

MONETARY MODEL

Under the rational expectations hypothesis, estimates of the forward spot exchange rate should be forecasted by the forward rate and an expectational error. The coefficient on the forward spot exchange rate should be one and under the REH, one would expect there to be no serial correlation among the errors. Yet, empirically REH appears to fail on all accounts. Often, researchers identify a severe downward bias associated with the estimated coefficient especially during shorter time horizons while other researchers appear to identify significant serial correlation among error terms.[16]

In light of the empirical evidence, researchers have called into question the validity of the REH. The forward premium puzzle may exists because of this violation, even under the assumption that capital is perfectly mobile. Additionally, the literature suggests that failures in econometric implementation can be an explanation for why the puzzle is present. This paper approaches both criticisms in attempts to explain the puzzle by including an alternative to rational expectations and a more robust set of econometric conditions which govern convergence and stability.

Bounded rationality as an alternative to the REH, was brought to the attention of macroeconomists by Sargent (1983) and formally presented by Evans and Honapohja. This paper treats participating economic agents in a similar vein, allowing agents to know the functional form of the model but do not know the parameter values which govern stability and learnability. Agents must act through their perceived parameter values and update their estimates as new information becomes available, essentially acting as applied econometricians. It is important to note, that market participants fail to anticipate regime shifts and only ex-post observe the regime. By assuming this, the transition matrix probabilities could be estimated econometrically through

[16] Research shows that empirically, the estimate slope coefficient is negative, even when adjusted for covered interest parity.

adaptive learning but for simplicity, I will provide a parameterization consistent with exchange rate behavior.

Unlike the assumptions provided by the REH, adaptive learning allows for the possibility of multiple equilibria. Moreover, AL can provide insight into which equilibrium is e-stable, and therefore coincides with the outcome found by RE. Also, by using the framework developed in an earlier paper, I am able to better analyze the dynamics of a speculative bubble solution.

Empirically, the existence of a bubble would provide evidence against the assumption of RE, considering the rational expectations framework believes an asset's value should reflect only its market fundamental value. Adaptive learning on the other hand, doesn't predispose this assumption but allows for the existence of a bubble. Moreover, AL and e-stability can be used to determine the learnability of multiple equilibria including that created from a bubble. Empirically, during the first half of the 1980s Meese (1986) supported the idea that the appreciation of the US Dollar (USD) in the currency market was due in part by a speculative bubble. The literature regarding bubbles and the rational expectations hypothesis within the exchange rate dynamics during this time are mixed. As reported by Wu (1995), exchange rate variability was found to be both caused and not caused by speculative bubbles depending on the author and the methodology.

Similar to the monetary models of Frenkel (1976) and Mussa (1976), I also introduces the monetary model as an appropriate representation of the exchange rate dynamics. For the sake of comparison, the monetary model first identifies the relationship between the exchange rate and its fundamental value, thought of as the long-run value of the exchange rate. Equation (2.1) depicts this difference,

$$\xi_t = f_t - s_t, \qquad (2.1)$$

where f_t represents the linear combination of relative money stock and real income between two countries at time *t*. s_t is considered the nominal exchange rate between a domestic and foreign country.[17] Thus, ξ_t represents the log difference in time *t* exchange rates from its long-run equilibrium value.

In order to evaluate the assumption of adaptive learning, equation (2.2) describes the k-period forecast of the change in exchange rates regressed against the current deviation of the exchange rate from its long-run fundamental value,

$$s_{t+k} - s_t = \alpha_k + \beta_k \xi_t + \nu_{t+k}, \qquad (2.2)$$

We would expect that under the rational expectations hypothesis, the regression coefficient from a simple OLS regression β_k should equal 1. Moreover, the regression coefficient allows us to compare the prediction power of short and long-run forecasts by analyzing the direction of β_k. We can assume that as the exchange rate under performs against its long-run average, the slope of the regression should be positive since the exchange rate should be mean reverting over time. Ultimately, this result is often found with a large enough time horizon. The literature suggests that the noise from short-run volatility averages out over time, so that prediction power increases.

The following framework of the monetary exchange-rate model is similar to what is used by Evans and Chakraborty (2008). The model assumes purchasing power parity, risk neutrality and uncovered interest parity. Equations (2.3)-(2.6) summarize the economy:

$$f_t = E_t s_{t+1}, \qquad (2.3)$$

$$i_t = i_t^* + E_t s_{t+1} - s_t, \qquad (2.4)$$

$$m_t - p_t = d_0 + d_1 y_t - d_2 i_t, \qquad (2.5a)$$

$$m_t^* - p_t^* = d_0' + d_1' y_t^* - d_2' i_t^* \qquad (2.5b)$$

$$p_t = p_t^* + s_t \qquad (2.6).$$

Equation (2.3) represents the risk neutral expectations of future market price of foreign currency, while equation (2.4) is the open parity condition with i_t and i_t^* representing domestic and foreign interest rates respectively. The money market equilibrium is found in equation (2.5a,b), where

[17] The variables f_t *and* s_t represent the logarithmic values

m_t is domestic money supply, p_t is the domestic price, and y_t is the level of domestic output. The final equation in the system represents the purchasing power parity condition, where p_t^* is foreign price level.

The parameters d_0, d_1, and d_2 are constant and not state-dependent. Evans and Chakroborty go on to also assume that the parameters are positive. This paper deviates from their methodology by assuming the parameters are state-dependent and follow a specific Markovian process. This assumption can potentially provide a more concrete reason as to the magnitude of short-run volatility generally found in these data sets. Economic agents must learn which regime they are experiencing, where under learning may update after multiple periods.

For this analysis, I assume that the parameter d_2 follows a two state Markov process. The parameter evolves according to the transition matrix,

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix},$$

where $P = (p_{ij}) for\ i, j = 1, 2$ with p_{ij} being the probability that $z_{t-1} = j$ given that $z_t = i$. I further assume that the transition matrix is non-absorbent, thus taken to be recurrent and aperiodic implying a unique stationary distribution.[18] The literature reveals that the parameter d_2, can be thought of as the interest semi-elasticity of money demand. More intuitively this parameter represents the sensitivity of money demand to changing interest rates. Moreover, given equation (2.4), d_2 ultimately reflects the difference in the fundamental value of the exchange rate from its current market value.

This system solves to yield the forward looking, non-linear, reduced form in equation (2.7)

$$\theta(z_t)s_t = \mu + d_2(z_t)E_t s_{t+1} + v_t. \qquad (2.7)$$

---

[18] This assumption has been outlined in Reed (2014)

Where the intercept term[19] $\mu = (d_0' - d_0)$, the coefficient $\theta(z_t) = 1 + d_2(z_t)$ and $v_t = (m_t - m_t^* - d_1(y_t - y_t^*))$ represents the fundamentals[20]. Furthermore, the fundamental component is assumed to follow an exogenous AR(1) process:

$$v_t = \rho v_{t-1} + \epsilon_t,$$

where the persistence factor ρ is close to one but is assumed to be $0 < \rho < 1$ and that ϵ_t represents white noise.

MEAN-SQUARE STABILITY

Evans and Chakraborty follow up the autoregressive assumption of the fundamentals by further assuming that the process v_t has compact support. This essentially guarantees the process exhibits finite moments of all orders. Essentially, this technical assumption ensures the exogenous process follows the theoretical learning results. This paper deviates from that assumption by adopting the solution method of Lubik and Schorfheide (2003, 2004).

Like Lubik and Schorfheide, I assume that the reduced form model represents the combination of the minimum state variable solution and the first-order moving average component which represents the determinate and indeterminate components of the solution respectively. Using the framework found in Farmer, Waggoner, and Zha (2009), I begin by postulating the MSV solution to (2.7) and solve through the method of undetermined coefficients

Suppose that a solution exits in the form

$$s_t = a(z_t)v_{t-1} + b(z_t)\epsilon_t \qquad (2.8)$$

where $a(z_t)$ and $b(z_t)$ represent the Markov switching rational expectations coefficients. Then it can be shown that a solution exists if,

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \theta_1 - p_{11}\rho & -p_{12}\rho \\ -p_{21}\rho & \theta_2 - p_{22}\rho \end{bmatrix}^{-1} \begin{bmatrix} \rho \\ \rho \end{bmatrix}$$

---

[20] Assuming similar parameter values for the foreign and home countries eliminates the constant term in the reduced form equation.

and

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{bmatrix}^{-1} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} \theta_1 - p_{11}\rho & -p_{12}\rho \\ -p_{21}\rho & \theta_2 - p_{22}\rho \end{bmatrix}^{-1} \begin{bmatrix} \rho \\ \rho \end{bmatrix}.$$

From the MSV solutions, the indeterminate solution or "bubble condition" can be expressed in terms of the moving average component ω_t and is shown to be dependent on the coefficient $(1 + \alpha(z_{t+1}, z_t)\theta(z_t))$. To see this first assume that the MSV can be rewritten to include the indeterminate component ω_t,

$$\omega_t = s_t - (a(z_t)v_{t-1} + b(z_t)\epsilon_t) \qquad (2.9)$$

and that by using Eq. (2.7) a solution to ω_t can be defined to be

$$\omega_t = \left(\theta(z_t)\right)^{-1} E_t \omega_{t+1}.$$

The solution can be rearranged in the expectation error form and becomes

$$\eta_{t+1} = \omega_{t+1} - \theta(z_t)\omega_t$$

where $E_t[\eta_{t+1}] = 0$. The expectational error equation yields,

$$\eta_{t+1} = \alpha(z_{t+1}, z_t)\theta(z_t)\omega_t + \beta(z_{t+1})(m\epsilon_{t+1} + \gamma_{t+1}) \qquad (2.10)$$

where $\alpha_{i,j}$ must satisfy the condition that $p_{11}\alpha_{11} + p_{21}\alpha_{21} = p_{12}\alpha_{12} + p_{22}\alpha_{22} = 0$ and *m* is any real number and γ_{t+1} is any i.i.d bounded stochastic process with mean zero and independence from other errors.

The solution to equation (2.10) can be given in the following form:

$$\omega_{t+1} = \left(1 + \alpha(z_{t+1}, z_t)\theta(z_t)\right)\omega_t + \beta(z_{t+1})(m\epsilon_{t+1} + \gamma_{t+1}).$$

Therefore we can see that the stability of the indeterminate solution depends on the eigenvalues of the coefficient matrix before ω_t. To be considered mean square stable and thus e-stable, the following definition from Farmer et al. applies

Definition 1 (mean square stable) A stochastic process x_t is mean square stable if there exist real numbers λ and ϕ such that

$$\lim_{s \to \infty} E_t[x_{t+s}] = \lambda$$

$$\lim_{s\to\infty} E_t[x_{t+s}^2] = \phi$$

Farmer et al. describe this condition as an important conclusion since it defines the existence of the second moments and thus allows for econometric testing to be done. In my earlier research I have shown that if a series is considered mean-square stable that the implication for learning is that economic agents would be able to learn the rational expectations solution. Furthermore, it is important to note that MSS is similar to the covariance stationary conditions set forth by Hamilton (1994) but generally are considered slightly weaker. Moreover for a constant-parameter, linear model, the conditions created by Blanchard and Kahn are equivalent to MSS.

In order to create more tractable stability conditions I assume that for the model state 2 is considered the indeterminate regime. Then a continuum of solutions exists for state 2 whereas state 1 has a unique solution. To begin, I define $\alpha(z_{t+1}, z_t)$ and $\beta(z_{t+1})$ in conjunction with the constraint on $\alpha(z_{t+1}, z_t)$:

$$\alpha_{11} = -1, \alpha_{12} = -1, \alpha_{21} = \frac{p_{11}}{p_{21}}, \alpha_{22} = \frac{p_{12}}{p_{22}}$$

and

$$\beta_1 = 0, \beta_2 = 1.$$

Assuming then that $p_{i,j} > 0 \; \forall \; i \; and \; j = 1{,}2$ then, ω_{t+1} is equal to

$$0 \; if \; z_t = 1 \; and \; z_{t+1} = 1$$

$$\frac{\theta_1}{p_{21}}\omega_t + (m\epsilon_{t+1} + \gamma_{t+1}) \; if \;\; z_t = 1 \; and \; z_{t+1} = 2$$

$$0 \; if \;\; z_t = 2 \; and \; z_{t+1} = 1$$

$$\frac{\theta_2}{p_{22}}\omega_t + (m\epsilon_{t+1} + \gamma_{t+1}) \; if \;\; z_t = 2 \; and \; z_{t+1} = 2.$$

Using the proof found in Farmer et al.: the series ω_{t+s} is bounded as $s \rightarrow \infty$, and thus determinate, i.f.f. $\left|\frac{\theta_2^2}{p_{22}}\right| < 1$.[21][22] It is worth noting that this definition of determinacy under mean-square stability coincides with the same definition presented in my earlier research. This result satisfies the thought that the spectral radius of the eigenvalues must be modulus one for mean square stability to occur.

By introducing this change, the influence of a bubble solution can be more thoroughly analyzed through mean-square stability. This sentiment is echoed closely by Farmer et al. who argue that indeterminacy can arise from the sunspot solution since the error term may create a non-stationary process when a unit root is present. In a monetary model, this qualitative feature is very important as changes in monetary policy can give rise to destabilization since shocks will be propagated through the series. This occurs because of the serial dependency found in many monetary variables including exchange rates.

To explore this further, the range of values that would prevent a bubble type solution can be identified by assuming the rate of persistency found in the second regime. Empirical estimations identify that the monetary fundamentals experience a two-state regime with each state showing a very high level of persistency. Therefore by assuming a persistency of 0.99, 0.95, 0.8, and 0.5 I find that interest semi-elasticity of money demand should be greater than 0.005, 0.02, 0.1, and 0.3 respectively. As I will show in the section, reasonable estimates place the interest semi-elasticity of money demand at values of .2 and .08 for quarterly data. For regime persistency levels below 0.95 this could point to why some bubble solutions could occur. This conclusion is an important result of this paper.

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[21] For the full proof see Farmer et al.

[22] This process can also be completed by assuming that state 1 is the indeterminate state and would garner a MSS condition of $\left|\frac{\theta_1^2}{p_{21}}\right| < 1$.

Table 1. Solutions for mean-square stability

$\left|\frac{\theta_2^2}{p_{22}}\right| < 1 \; and \; \left|\frac{\theta_1^2}{p_{21}}\right| < 1$

	$p\ when\ s_{t+1} = 2\ and\ s_t = 2$			
	0.99	0.95	0.8	0.5
$d_2(2) >$	0.005	0.02	0.1	0.3
	$p\ when\ s_{t+1} = 2\ and\ s_t = 1$			
	0.1	0.25	0.5	0.6
$d_2(1) >$	0.7	0.5	0.3	0.2

Note that only the real numbers are reported as the absolute value produces imaginary numbers

ADAPTIVE LEARNING

Agents are assumed to form the expected future exchange values by first generating their Perceived Law of Motions (PLM). This perception begins with an estimate of the MSV solution found in Eq. (2.8) and (2.9) creating an equation with the following form:

$$s_t = a(z_t)v_{t-1} + b(z_t)\epsilon_t + \omega_t$$

where ϵ_t is white noise stemming from the observed fundamentals and *a* and *b* are considered the rational expectations estimates. Since agents are expected to know the parameters but not necessarily the model form, RE produces Eq. (2.11):

$$s_t = \bar{a}(z_t)v_{t-1} + \bar{b}(z_t)\epsilon_t + \omega_t, \qquad (2.11)$$

where $\bar{a}(z_t)\ and\ \bar{b}(z_t)$ are defined by the solutions found in the previous section.

As previously expressed the key feature of adaptive learning is that agents have imperfect knowledge regarding the parameter values $\bar{a}(z_t)\ and\ \bar{b}(z_t)$. The timing of the observation becomes critical for understanding agents' ability to forecast future spot exchange rates. At the end of period *t-1* agents observe and collect the actual parameter values so that in time *t*, they are able to use the history of the parameters up to and including *t-1* in order to make their estimation of s_{t+1}. After the period *t* ends agents observe their forecast error and use it that to update the estimated coefficients for *t+1.* Agents continue this process until their estimates converge with the RE estimate or diverge and produce an alternative solution. What dictates the mapping of the PLM to the Actual Law of Motions (ALM), that is the true process the series follows, is the e-stability condition. Again, e-stability refers to agents learning the rational expectations solution. It has been noted by Chakraborty (2009) and Evans and Honkapohja (2001) that different learning algorithms will produce a variety of learning behavior and not all will be e-stable. For this analysis I rely on the algorithm of constant gain learning. Agents assumed to learn this way are very sensitive to the forecast error observed at the end of each period. This is due to the fixed nature of the sensitivity rather than a gradual decline of the sensitivity value found in traditional recursive least squares learning. Moreover, Branch and Evans (2008) demonstrate that constant gain learning can influence the cycle of bubbles and crashes found in some asset pricing. This result underlines an important outcome found in all adaptive learning processes, that is these beliefs inherently create serial correlation which may not exists otherwise.

Sargent describes this process of transitioning from an uncorrelated process to a serially correlated one via a random walk forecasting model. Intuitively, agents begin to track their own serial correlation and are able to produce recurrent bubbles because of their beliefs. Marcet and Nicolini (2003) and Sargent, Williams, and Zha (2006b) are able to create the recurrent bubbles and crashes in their models of hyperinflation and credit constant gain learning for the result. They find that bubble type hyperinflationary paths are unstable under constant gain learning. For recent monetary models, featuring constant gain learning has become very popular. The linear forecasting rule is shown to be similar to the reduced-form rational expectations solution. The self-referential nature of constant gain learning makes persistence escapes from the rational expectations estimates a very plausible possibility. This is due in part by the emphasis on recent forecast errors. Every period, agents use a constant parameter, which remains unchanged over time, in order to capture the sensitivity to forecast errors. To see this I begin by defining the parameter and shock matrices, $X_t = \binom{1}{v_t}$ and $\Omega_t(z_t) = \binom{a_{t,z_t}}{b_{t,z_t}}$ respectively. Then by using the Recursive Least Squares algorithm, the estimates are produced from the following system of equations:

$$\Omega_{t,z_t} = \Omega_{t-1,z_{t-1}} + \gamma S_{t-1}^{-1} X_{t-1}(X_t - \Omega'_{t-1,z_{t-1}} X_{t-1}) \qquad (2.12)$$

$$S_t = S_{t-1} + \gamma(X_t X_t' - S_{t-1}). \qquad (2.13)$$

Agents use Eq. (2.12) to update their estimates. The gain factor is represented by $\gamma S_{t-1}^{-1} X_{t-1}$ where γ the parameterized sensitivity or constant gain recognized by the agents each period. Last period's estimates are designated by $\Omega'_{t-1,z_{t-1}}$ and the forecasting error is defined by $X_t - \Omega'_{t-1,z_{t-1}} X_{t-1}$. In order for agents to obtain an estimate of the exchange rate in period *t*, they must first forecast the *t+1* exchange rate. Agents in time *t* use values of the parameters in *t-1* to estimate time *t* values using Eq. (2.12) and (2.13). Since agents' forecasts are dependent on the information in the previous time period, an unannounced and unobserved regime change would render the previous information inadequate for parameter estimation. Agents would have to observe enough new regime outcomes in order to appropriately assign the importance on historical parameter observations. Intuitively agents' parameter estimates will converge slower to the RE outcomes as well as provide the necessary shock so that agents follow a self-referential path of estimation. Thus, agents who are assumed to be bounded in their rationality and therefore required to estimate parameter values through constant gain learning while regime changes are occurring may be able to better match the empirical data and offer an explanation for the forward premium puzzle.

PARAMETERIZATION AND SIMULATION

Chakraborty and Kim attempt to include regime changes in their respective analysis by creating an artificial structural break in the simulation of parameter values. Chakraborty parameterizes the intercept term in the law of motions equation to evolve according to a two-state Markov process. This type of break was chosen arbitrarily whereas this paper employs empirical estimates of the evolution of the Markov process and includes them in the learning process. Furthermore, both Kim and Chakraborty fail to recognize the importance of the regime spillover created by the state dependent parameters. I have shown that parameter values governing expectations on the future spot exchange rate now depend on the probability of transitioning out of the current regime, a feature lacking in previous research.

To estimate values for the parameter choices, I use data found on the St. Louis FRED database as well as the Bank of England and finally found through Bloomberg. Monthly values were procured starting in January, 1989 and ending in October, 2014 producing a sample size of 309 observations. The monetary fundamentals were created by using monthly M1 estimates for Switzerland (CHF), Japan (JPY), U.K. (GBP), and the U.S. (USD). Daily exchange rates, with the USD being the numeraire currency, were averaged to create monthly values. Monthly forward exchange rates looking out over 3-months were taken from the Bank of England and from archived data from Bloomberg. Quarterly real gross domestic product was used to create monthly estimates using a cubic spline interpolation. Newey-West standard errors were used for the OLS regressions to account for the autocorrelation created by the spline interpolation.

The parameter choices for $\theta(z_t), \rho, and\ \gamma$ play an important role for the development of the stochastic process. The interest semi-elasticity of money demand, d_2, is the parameter that I have chosen to switch regimes. Intuitively, this parameter is sensitive to adjustments made to the interest rate by central banks and thus could garner different values. This is contrary to the approach taken by Chakraborty, Evans and Chakraborty, and Kim. Each study assumes that the interest semi-elasticity value to be -0.08 for quarterly data and cite the parameterization to be taken from Stock and Watson (1993) who find the value for annual data to be -0.02. Yet according to Ball (2001), the interest semi-elasticity of money demand for annual data tends to be -0.05, meaning that for quarterly data the value should be around -0.2. Assuming that the value for d_2 is the negative of interest semi-elasticity of money demand and in terms of basis points value, $\theta(z_t) = (1 + d_2(z_t))^{-1}$, when $d_2(2) = 20$ will be 0.048 and when $d_2(1) = 8$, θ becomes 0.11.

Moreover, offering no intuition, this body of literature assumes that the structural change will occur in the intercept term of the fundamentals equations to be estimated. Chakraborty finds mixed results when searching for a break using a variety of tests. He concludes that the possibility of a structural break appears plausible in the exchange rate series for JPY/USD and the GBP/USD therefore supporting the use of constant gain learning. I find evidence of a structural break occurring by running a Markov-switching dynamic regression. By allowing the constant term as well as the persistence parameters attached to the AR(1) and AR(2) components of the JPY/USD and GBP/USD exchange rates to be state-dependent I find significant evidence of a two-regime model. The SBIC value confirmed that a two-state model was preferable to a three-state. As expected, the CHF/USD exchange rate was modeled best by a one-regime model corroborating the pegged exchange rate regime Switzerland has historically experienced.[23]

Furthermore, the value for the gain parameter has inherent implications for agents learning the rational expectations solutions. Branch and Evans indicate that common values found for γ range from 0.01 to 0.1, where larger values for the gain parameter often lead to escapes from the RE solution. Whereas Orphanides and Williams (2005) suggest that for quarterly data a gain value of 0.02 is ideal. I choose values in this range to observe the effect on escaping the RE solution and producing a bubble like outcome. This outcome is further suggested by Kim who uses constant gain learning and a shift in fundamental values as the basis for a structural break. This paper uses the fundamental process as the basis for a structural change but again emphasizes that this regime shift happens in the parameter of d_2.

Additionally, Kim finds that the persistent nature of the fundamental process is contingent on introducing structural breaks into the series. Like Kim, I find that the value of ρ is shown to be highly persistent in the one-regime case for all exchange rate series but only the USD/JPY identified a unit root. As state-dependent parameters were introduced, the first state for the

[23] For the full set of figures and tables please see Appendix A.

USD/GBP fundamentals exhibited high levels of persistency but was significantly less than one. All fundamental values for the second state failed to reject the null hypothesis that ρ=1. Table 2 captures the effect that the regimes have on the fundamental components. The monetary fundamentals for the Swiss Franc exemplifies the change in persistence as it now fails to be significantly lower than unity.

Table 2. Monetary Fundamentals AR(1)

$$v_t = \alpha(z_t) + \rho(z_t)v_{t-1} + \epsilon_t$$

		GBP	JPY	CHF
	ρ	0.995* (0.003)	0.997 (0.001)	0.998* (0.003)
	α	0.017 (0.014)	-0.001 (0.001)	-0.006 (0.009)
	R^2	0.99	0.97	0.99
State 1	ρ	0.997* (0.001)	1.003** (0.000)	0.999 (0.001)
	α	4.760* (0.414)	11.397 (75.046)	-23.330 (64.701)
	p_{11}	0.99 (0.004)	0.99 (0.005)	0.95 (0.028)
State 2	ρ	1.072 (0.050)	1.003 (0.021)	1.000 (0.000)
	α	4.85 (0.413)	11.397 (75.046)	-23.330 (64.701)
	p_{22}	0.59 (0.199)	0.63 (0.170)	0.99 (0.011)

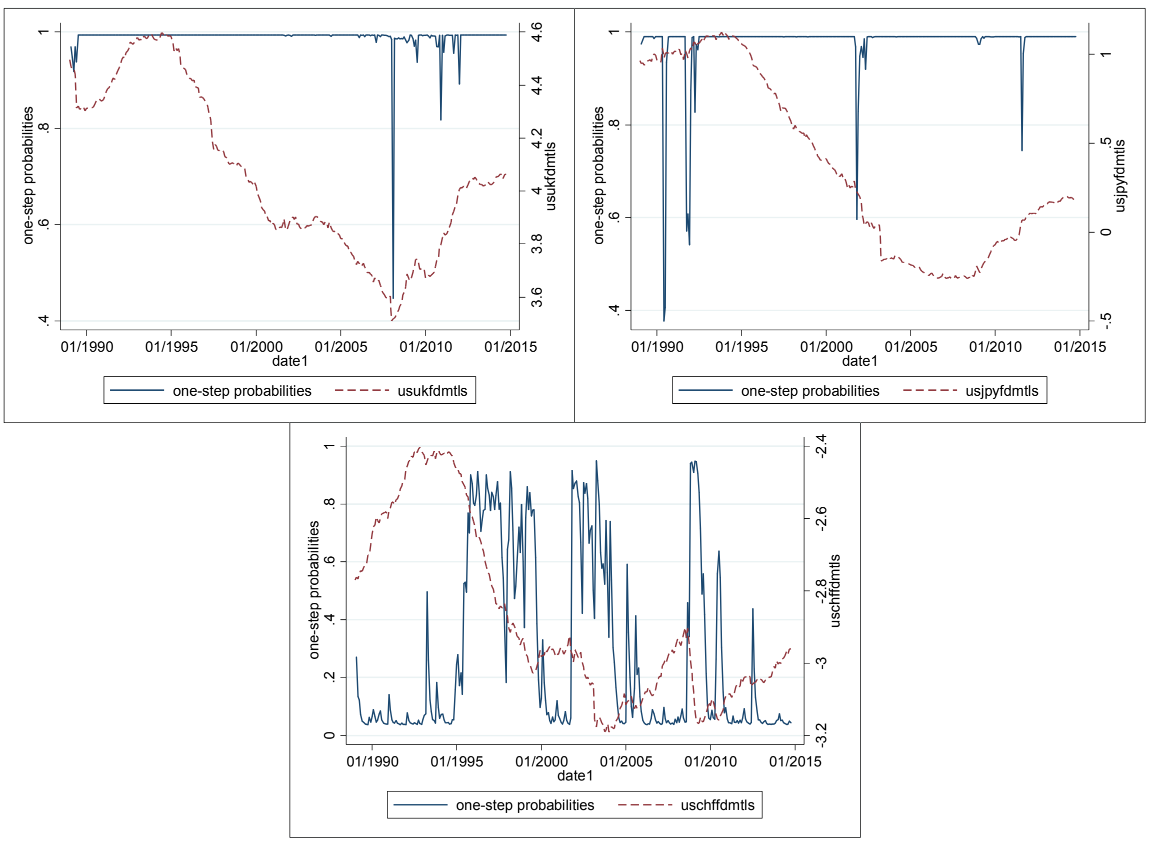
Note that * denotes significance at the 95% level, where H_0: ρ=1 and H_0: α=0. Also, the notation ** identifies a value significantly greater than 1.

This outcome supports the argument found in Chakraborty that highly persistent fundamental series will often contribute to the negative direction of the estimated coefficient on the forward premium. He further concludes that as the persistence declines, the negative sign found from the regression on the premium tends to vanish.

As for each fundamental series the SBIC criterion suggests that the structural breaks follow a two-state Markov process which was preferable to a three-state process. Moreover Figure 1 below compares the transition probabilities for each monetary fundamentals series.

The transition probabilities suggest that no state for each fundamental series is absorbing. It is also noted that no state has consistently higher variance among the fundamentals. This is a different result from Kim, who found that some regimes exhibited a significantly higher variance than other regimes.

Figure 1. Transition Probabilites and Monetary Fundamentals for GBP/USD, CHF/USD, and JPY/USD



These observations lead to an important conclusion regarding the use of constant gain parameter. Recent research suggests that constant gain learning properly represents agents beliefs when they believe structural shifts can occur. Studies from Bullard and Eusepi (2005), Sargent and Williams (2005) among others intuitively argue that agents will place a considerable amount of weight on recent data when continuous structural shifts are apparent.

To exemplify the forward premium puzzle, Table 3 below outlines the regression of monthly depreciation on the 3-month forward premium. As Fama expressed, the coefficient on the forward premium theoretically should be one but often shows significant deviations from this value. The coefficients that I find are typical of this literature. For each series, the coefficients are significantly lower than unity and negative for the CHF/USD and JPY/USD exchange rate premiums. Researchers continue to offer explanations for this phenomenon, with adaptive learning now taking center stage. In order to explain the deviation from unity coupled with a negative direction, I create a simulated learning structure that adheres to the assumption that agents have a difficult time estimating the theoretical parameters under constant gain learning when facing persistent structural breaks.

Table 3. Regression of monthly depreciation on 3-month forward premium

$$s_{t+1} - s_t = \alpha + \beta(f_t - s_t) + \mu_{t+1}$$

	GBP/USD	CHF/USD	JPY/USD
$\hat{\alpha}_{\text{ols}}$	-0.007	-0.003*	-0.002*
	(0.050)	(0.001)	(0.001)
$\hat{\beta}_{\text{ols}}$	0.001*	-0.446*	-0.497*
	(.011)	(0.071)	(0.066)
R^2	0.0004	0.11	0.16
n=	309	309	309

Note that * denotes signifiance at the 95% level, where for $H_{0:}\ \alpha=0$ while for $H_{0:}\ \beta=1$.

Furthermore, I estimate the Markov switching maximum likelihood estimate for β and find similar results to the single regime OLS estimates. Table 4 shows that the forward premium regression for the GBP/USD results in very different regime coefficients with one state failing to be statistically different from unity, whereas JPY/USD displays quantitatively similar parameter estimates to the single regime OLS. It should be noted that due to the Swiss exchange rate regime persistence, multiple regime could not be estimated using standard maximization techniques.

Table 4. Dynamic Markov Regression of monthly depreciation on 3-month forward premium

$s_{t+1} - s_t = \alpha(z_t) + \beta(z_t)(f_t - s_t) + \mu_{t+1}$

	GBP/USD	JPY/USD
$\widehat{\alpha_1}_{\text{MLE}}$	-0.025	-0.055*
	(0.057)	(0.009)
$\widehat{\alpha_2}_{\text{MLE}}$	-0.262	-0.001
	(0.314)	(.002)
$\widehat{\beta_1}_{\text{MLE}}$	0.004*	-0.538*
	(.013)	(0.292)
$\widehat{\beta_2}_{\text{MLE}}$	0.692	-0.460*
	(0.692)	(0.066)
n=	309	309

Note that * denotes signifiance at the 95% level, where for $H_{0:}$ α=0 while for $H_{0:}$ β=1.

This exercise in parameter estimation is encouraging as it isolates the importance of regime switching for at least one of the series being analyzed. It provides further evidence for using state dependent parameters in conjunction with constant gain learning.

Similar to the setup found in Evans and Chakraborty, I simulate the forward premium regression under the assumption that economic agents are using constant gain learning to estimate the coefficient on the forward premium. I begin by creating 1000 simulations using a fairly large sample size, *T*=20,000 over a range of gains $\gamma > 0$. I discard the first 20,000 data points and report the mean value from the remaining sample across simulations. The simulations were done with and without an intercept term. Table 5 presents the comparison between simulated values for $\hat{\beta}_{sim}$.

Table 5. Results from 1000 simulations of the model for sample size 360 with regime change

	$\theta_1 = 8, \theta_2 = 20$	ρ	γ	no intercept avg. $\hat{\beta}_{sim}$	t-stat	intercept avg. $\hat{\beta}_{sim}$	t-stat
	8,20	0.95	0.1	0.34	-3.69	0.36	-3.53
	8,20	0.95	0.05	0.47	-2.83	0.50	-2.58
	8,20	0.95	0.03	0.52	-2.63	0.56	-2.28
	8,20	0.95	0.02	0.53	-2.51	0.57	-2.24
	8,20	0.95	0.01	0.53	-2.63	0.58	-2.23
	8,20	0.95	0.001	0.55	-2.55	0.62	-2.00
	8,20	0.995	0.1	0.11	-7.06	0.22	-4.54
	8,20	0.995	0.05	0.13	-6.75	0.27	-4.03
p11=.95,	8,20	0.995	0.03	0.13	-6.78	0.30	-3.79
p22=.95	8,20	0.995	0.02	0.13	-6.93	0.30	-3.83
	8,20	0.995	0.01	0.13	-7.08	0.30	-3.92
	8,20	0.995	0.001	0.14	-7.18	0.30	-3.99
	8,20	0.999	0.1	0.04	-14.55	0.20	-4.75
	8,20	0.999	0.05	0.04	-13.65	0.25	-4.31
	8,20	0.999	0.03	0.05	-13.25	0.26	-4.22
	8,20	0.999	0.02	0.04	-13.69	0.27	-4.16
	8,20	0.999	0.01	0.04	-13.96	0.27	-4.15
	8,20	0.999	0.001	0.04	-14.82	0.25	-4.39

It is apparent that with the parameter choices assumed from the previous section, learning simulates the data very well. With moderately persistent regimes, the gain parameter fails to account for the theorized value of the coefficient. Moreover, as the regimes becomes more persistent I am able to show that agents would be able to learn the theorized value of premium coefficient. This is an important finding. Even though the persistence of the fundamentals shows that a unit root may be present in the process, highly persistent fundamentals can still result in a coefficient similar to the theoretical value of one. This appears to be due in part to the persistence of the regimes. These findings are somewhat complementary to that of Evans and Chakraborty. They express the intuition behind this phenomenon is that when agents are faced with perpetual shock to the fundamentals, essentially when $\rho = 1$, agents must track the progress more closely with the gain parameter and this equates to using a larger gain. This was not the case though when the fundamentals do not need to be accounted for so intently. This paper finds that regime persistence plays an important role in the formulation of the premium coefficient estimates. I find that when $\rho = 1$, and when regimes have low persistence, agents will not achieve the unity estimate found by Chakraborty. In fact, the estimates match very closely to the empirical estimate for the JPY/USD forward exchange regression. Overall, regardless of the persistence agents prefer using a smaller gain parameter. Intuitively this behavior is similar to placing smaller and smaller importance on past observations. I believe this is due solely to the introduction of regimes into the learning analysis and is a crucial finding of this paper.

Table 6. Results from 1000 Simulations of the Model for Sample Size 360 with Regime Change: Comparison of Regime Persistence

	$\theta_1 = 8, \theta_2 = 20$	ρ	γ	No Intercept, median $\hat{\beta}_{sim}$	t-stat	Intercept, median $\hat{\beta}_{sim}$	t-stat
p11=0.9, p22=0.9	8,20	0.999	0.1	0.021	-30.51	0.128	-9.92
p11=.99, p22=.95	8,20	0.95	0.01	0.8066*	-0.65	0.9014*	-0.31
p11=.99, p22=.99	8,20	0.95	0.01	0.8091*	-0.67	0.8815*	-0.40
p11=.99, p22=.99	8,20	0.95	0.1	0.200	-3.55	0.204	-3.49
p11=.999, p22=.999	8,20	0.95	0.01	0.907*	-0.27	0.9882*	-0.03
p11=.999, p22=.999	8,20	0.95	0.1	0.105	-3.08	0.106	-3.76
p11=1, p22=1	8,20	0.9	0.01	0.9623*	-0.17	1.0117*	0.05
p11=1, p22=1	8,20	0.95	0.01	0.9159*	-0.25	0.9996*	0.00
p11=1, p22=1	8,20	0.95	0.1	0.121	-3.81	0.123	-3.78
p11=1, p22=1	8,20	0.99	0.01	0.4366*	-0.76	0.5492*	-0.57

Note that * denotes significance at the 95% level.

Table 7. Results from 1000 Simulations of the Model for Sample Size 360 with Regime Change: Comparison of Fundamental Persistence

	$\theta_1 = 8, \theta_2 = 20$	ρ	γ	No Intercept, median $\hat{\beta}_{sim}$	t-stat	Intercept, median $\hat{\beta}_{sim}$	t-stat
p11=0.9, p22=0.9	8,20	1	0.1	0.000	-167.39	0.062	-15.62
p11=0.9, p22=0.95	8,20	1	0.1	0.002	-88.21	0.105	-7.68
p11=0.95, p22=0.95	8,20	1	0.01	0.002	-84.60	0.129	-7.09
p11=0.99, p22=0.99	8,20	1	0.01	0.005	-17.91	0.419*	-1.00
p11=1, p22=1	8,20	1	0.01	-0.084	-1.59	-0.412	-1.81
p11=1, p22=1	8,20	1	0.1	-0.040	-4.31	-0.098	-4.44
p11=1, p22=.99	8,20	1	0.1	-0.034	-4.25	-0.079	-4.36

Note that * denotes significance at the 95% level.

CONCLUSIONS

The forward premium puzzle has seen a variety of macroeconomic methodologies introduced in order to explain away the empirical results that Fama first observed. Recently, adaptive learning literature from Evans and Chakraborty, Chakraborty, and Kim, has explored the role that recursive least squares plays in explaining the forward premium puzzle. As it happens, adaptive learning does offer an explanation of many of the key empirical characteristics outlined by Fama.

This paper contributes to the body of literature by introducing state dependent parameters governed by Markov switching regimes. Because of the complications that occur due to the spillover effects from changing regimes, traditional expectational stability analysis needs to be modified. These features impart a key distinction between this paper and the previous body of research. Furthermore, this paper assumes that agents learn using the method of constant gain rather than decreasing gain.

Constant gain learning provides agents with a plausible method to moderate their error when forecasting the forward exchange rate. Moreover, by parameterizing the model with a constant term instead of one that decreases over time, I can simulate the differences in perceived gain to determine the empirical importance. I believe that constant gain learning is necessary in this macroeconomic climate as it lends itself to persistent data series, which is observed in both the monetary fundamentals and exchange rate series. It should also be mentioned that constant gain learning can be a contributing factor of deviations from the rational expectations solution.

Deviating from the canonical literature, mean square stability captures the competing regime effects, noted by Farmer et al. and produces tangible conditions to measure e-stability. I find that there is a very distinct possibility that the regimes trigger an indeterminate bubble solution even under robust parameterization. This is an important result, as it points toward agents failing to reach the rational expectations solutions under the assumption of regime dependent parameters. Providing further evidence for why the forward premium puzzle exists.

Empirically, the coefficient from the forward premium OLS regression shows the standard signs of deviating from the theoretical value of unity. The downward bias, so thoroughly analyzed by Evans and Chakraborty, is present in the empirical estimates. Furthermore, this paper uses MLE techniques to observe the regime changes in the exchange rate series. As expected, the regimes produce slightly different results than what has been reported historically for single regime regressions. Nonetheless, both estimates produce the observed forward premium puzzle. As for the estimated parameter values, I find that the monetary fundamentals series is extremely persistent but not divergent under a single regime for the GBP/USD and CHF/USD series, but found an unstable eigenvalue for the JPY/USD series. Estimating the persistence under the assumption of two-regimes produces a unit root for all three series in the second state and for JPY/USD and CHF/USD in the first state. Moreover, I found that a two-state regime was sufficient for estimating the monetary fundamentals and the exchange rate series. The two states were highly persistent but not absorbing. These estimates provided support for the parameterization required for simulating the forward premium regression under constant gain learning.

The simulation results under state-dependent constant gain learning revealed the importance of both the persistency of the fundamentals and the regimes. The first result corroborated the results found common to recent learning literature but the second result remains a unique finding and an important one to this research. Moreover, I am able to simulate a premium coefficient which matches the empirical findings by using a parameterization estimated from the data. This finding points toward the importance of using constant gain adaptive learning to explain the forward premium puzzle.

Furthermore, I was able to simulate the theoretical value of the premium coefficient by adjusting the persistence of the regimes even under highly persistent values for the fundamental series. This again points to the importance of not only the fundamental series but of the possibility of regime changes. Under the presence of regimes, simulations revealed that smaller constant gain parameters performed better than larger gain values. This conclusion seems to be a product of agents anticipating regime changes, therefore preferring to place lower emphasis on current estimation errors.

The results of this paper conclude that in conjunction with the constant gain adaptive learning approach state-dependent parameters which follow Markov switching transition matrix should be a used in future empirical applications of the forward premium puzzle.

APPENDIX A

Proof for Lemma 1

Let y_t be a solution of the stacked system. By substituting the moving average component into the stacked system and using the definition of G_{s_t},

It follows that the process w_t is a solution of

$$\Gamma_{s_t} w_t = E_t[w_{t+1}].$$

Let V_i be any matrix with orthonormal columns s.t. the column space of V_i is the span of the support of $w_t 1_{\{s_t=i\}}, where\ 1_{\{s_t=i\}}$ denotes the indicator function that is one if $s_t = i$ and zero otherwise. Let k_i be the dimension of the column space of V_i. Since it is shown that w_t is a solution to the stacked system, the following equation holds almost surely, with a probability limit of one.

$$\Gamma_i v = E\left[\Gamma_{s_t} w_t \middle| w_t = v, s_t = i\right] = E[E_t[w_{t+1}]|w_t = v, s_t = i]$$

$$= E[w_{t+1}|w_t = v, s_t = i] = \sum_{j=1}^{h} p_{ij} E[w_{t+1}|w_t = v, s_t = i, s_{t+1} = j].$$

Because the column space of V_j is the span of the support of $w_{t+1} 1_{\{s_{t+1}=j\}}$, it follows that $E[w_{t+1}|w_t = v, s_t = i, s_{t+1} = j]$ is almost surely in the column space of V_j. This and the fact that the column space of V_i is the span of the support of $w_t 1_{\{s_t=i\}}$, implies that there exists a $k_j\ x\ k_i$ matrix Φ_{ij} such that

$$\Gamma_i V_i = \sum_{j=1}^{h} p_{ij} V_j\, \Phi_{ij}.$$

Define $\gamma_t = w_t - V_{s_t} \Phi_{s_{t-1}, s_t} V'_{s_{t-1}} w_{t-1}$. Because w_t, and hence γ_t, is almost surely in the column space of $V_{s_t}, \gamma_t = V_{s_t} V'_{s_t} \gamma_t$. The last remaining component of the proof is to show $E_{t-1}\left[V_{s_t} V'_{s_t} \gamma_t\right] = 0$. Since

$$E_{t-1}\left[V_{s_t} V'_{s_t} \gamma_t\right] = E_{t-1}[w_t - V_{s_t} \Phi_{s_{t-1}, s_t} V'_{s_{t-1}} w_{t-1}]$$

$$= \Gamma_{s_{t-1}} w_{t-1} - \sum_{j=1}^{h} p_{s_t,j} V_j \Phi_{s_{t-1},j} V'_{s_{t-1}} w_{t-1}$$

$$= \Gamma_{s_{t-1}} w_{t-1} - \Gamma_{s_{t-1}} V_{s_{t-1}} V'_{s_{t-1}} w_{t-1}$$

$$= 0.$$

Where the last equality holds because w_{t-1} is almost surely in the column space of $V_{s_{t-1}}$.

Proof of Corollary 1

The original proof follows almost directly from Costa et al. but for certain timing conventions, I present the proof from Farmer et al. with slight modifications.

From Lemma 1, y_t has been shown to be solution of the stacked system. Following the results of determinacy, one must show that the process, y_t is MSS if and only if the $r_\sigma\left(M_1(\Phi_{ij})\right) \geq 1$. Since I have defined the shock process to be mean-zero and independent of the state, s_t, for any y_t it must be that

$$E[y_t] = E[w_t] \; and$$

$$E[y_t y_t'] = E\left[G_{s_t} u_t u_t' G_{s_t}'\right] + E\left[G_{s_t} u_t \gamma_t' V_{s_t}' V_{s_t}\right] + E\left[V_{s_t} V_{s_t}' \gamma_t u_t' G_{s_t}'\right] + E[w_t w_t'].$$

Continue to let u_t and γ_t to be MSS and independent of s_t, it follows that the first three terms will converge as *t* increases without bound. Thus y_t will be MSS if and only if w_t is MSS.

To show that w_t is MSS, I first appeal to the result that w_t is MSS if and only if $r_\sigma\left(M_1(\Phi_{ij})\right) < 1$. This can be done by showing that the $\lim_{t\to\infty} E[w_t w_t']$ exists if and only if $M_1(\Phi_{ij}) < 1$ and then showing that if $M_1(\Phi_{ij}) < 1$ this implies $\lim_{t\to\infty} E[w_t]$ exists. Note that

$$E\left[w_t w_t' 1_{\{s_t=j\}}\right] = \sum_{i=1}^{h} p_{ij} \Lambda_{ij} E\left[w_{t-1} w_{t-1}' 1_{\{s_{t-1}=j\}}\right] \Lambda_{ij}' + V_j V_j' E[\gamma_t \gamma_t' 1_{\{s_t=j\}}] V_j V_j'.$$

Since we have defined γ_t to be mean-zero and independent of the Markov process it follows that $E\left[\left(\gamma_t 1_{\{s_t=j\}}\right)\left(w_{t-1}' 1_{\{s_{t-1}=j\}}\right)\right] = 0.$

Define linear operators

$$T(X_1,\dots,X_h) = (\sum_{i=1}^{h} p_{i1}\Lambda_{i1}X_i\Lambda_{i1}', \dots, \sum_{i=1}^{h} p_{ih}\Lambda_{ih}X_i\Lambda_{ih}'),$$

$$S(X_1,\dots,X_h) = X_1 + \dots + X_h,$$

where X_j is an $n\ x\ n$ matrix. Let

$$\Sigma_{j,t} = E\left[w_t w_t' 1_{\{s_t=j\}}\right],$$

$$\Sigma_t = \left(\Sigma_{1,t},\dots,\Sigma_{h,t}\right),$$

$$\Omega_{j,t} = V_j V_j' E\left[\gamma_t\gamma_t' 1_{\{s_t=j\}}\right] V_j V_j',$$

$$\Omega_t = \left(\Omega_{1,t},\dots,\Omega_{h,t}\right).$$

Since γ_t is MSS, $\lim_{t\to\infty} E[\gamma_t\gamma_t'] = \Omega_\gamma$ for some symmetric and positive semi-definite matrix Ω_γ. It follows from the definition of γ_t that

$$\Omega = \lim_{t\to\infty}\Omega_t = \left(p_1 V_1 V_1'\Omega_\gamma V_1 V_1',\dots,p_h V_h V_h'\Omega_\gamma V_h V_h'\right),$$

where p_j is the ergodic probability that $s_t = j$. By iterating the linear operators I find

$$\Sigma_t = T^t(\Sigma_0) + \sum_{k=1}^{t} T^{t-k}\Omega_k.$$

Also, since *T* is a linear operator, its matrix representation is given by

$$vec\left(T(X_1,\dots,X_h)\right) = M_1\left(\Lambda_{ij}\right)vec(X_1,\dots,X_h),$$

where the *vec* operator stacks the columns of the matrices into a column vector. So, if $r_\sigma\left(M_1(\Phi_{ij})\right) < 1$, then

$$\lim_{t\to\infty} E[w_t w_t'] = \lim_{t\to\infty}\sum_{k=1}^{t} T^{t-k}\Omega_k = \sum_{k=0}^{t} T^k(\Omega),$$

where the last term is a convergent series. The other scenario which needs to be addressed is if w_t is MSS so that $\lim_{t\to\infty} E[w_t w_t']$ exists and does not depend on the initial condition Σ_0, then it must be the case that $\lim_{t\to\infty} T^t(\Sigma_0) = 0$ for any $\Sigma_0 = (\Sigma_{1,0},\dots,\Sigma_{h,0})$ such that $\Sigma_{j,0}$ is symmetric and positive semi-definite. Branch et al. reveal that if $\lim_{t\to\infty} T^t(\Sigma_0) = 0$ for any $\Sigma_0 = (\Sigma_{1,0},\dots,\Sigma_{h,0})$ such that $\Sigma_{j,0}$

is symmetric and positive semi-definite, then it is the case that $\lim_{t\to\infty} T^t(\Sigma_0) = 0$ for any Σ_0, which implies that $r_\sigma\left(M_1(\Phi_{ij})\right) < 1$. This is because

$$M_1(\Lambda_{ij}) = diag(V_i \otimes V_i) M_1(\Phi_{ij}) diag(V_i' \otimes V_i'),$$

The non-zero eigenvalues of $M_1(\Lambda_{ij})$ are the same as the non-zero eigenvalues of $M_1(\Phi_{ij})$, which implies the spectral radius of Λ_{ij} is equal to the spectral radius of Φ_{ij} or,

$$r_\sigma\left(M_1(\Phi_{ij})\right) = r_\sigma\left(M_1(\Lambda_{ij})\right).$$

Thus it has been shown that $\lim_{t\to\infty} E[w_t w_t']$ exists if and only if $M_1(\Phi_{ij}) < 1$.

Proof of Lemma 2

Let y_{it} identify as an RDE. Let $f_t = (y, s|s_{t-1} = i, \Omega_{t-1})$, where f_t is the joint density of y_t and s_t conditional on $s_{t-1} = i$ and on all other time *t*-1 information, not including current and past s_{t-1}, included in Ω_{t-1}. Also, let $f_t^i = (y|\Omega_{t-1})$ be the density for y_{it} conditional on Ω_{t-1}, and $f_t = (s|s_{t-1} = i)$ be the conditional density of s_t given $s_{t-1} = i$. Where $f_t(s = j|s_{t-1} = i) = P_{ij}$. Expectations can be computed then, as follows:

$$E(y_{t+1}|s_t = i, \Omega_t) = \iint y f_{t+1}(y, s|s_t = i, \Omega_t)\, dsdy$$

$$= \iint y f_{t+1}(y|s, s_t = i, \Omega_t)\, f(s|s_t = i) dsdy$$

$$= \iint y f_{t+1}^s(y|\,\Omega_t)\, f(s|s_t = i) dsdy$$

$$= \textstyle\sum_j P_{ij} E_t y_{jt+1}.$$

This representation of expectations for y_t can be used to make sure the stacked system is satisfied.

To ensure E-stability, Branch et al. construct a proposition which uses the forward looking expectational difference equation, similar to the stacked system. They conclude that MSV forecasting using an adaptive learning algorithm, like least squares, will converge to the unique RDE and this be E-stable.

Proof of Lemma 3

Given the PLM $y_t = A(s_t) + B(s_t)r_t$, expectations are state contingent, where $s_t = j$ for all $j = 1, \ldots, m$ implies

$$E_t(y_{t+1}|s_t = j) = p_{j1}A(1) + p_{j2}A(2) + \cdots + p_{jm}A(m) + \left(p_{j1}B(1) + p_{j2}B(2) + \cdots + p_{jm}B(m)\right)\rho r_t$$

The state-contingent ALM or T-map is then

$$A(j) \to \beta_j(p_{j1}A(1) + p_{j2}A(2) + \cdots + p_{jm}A(m))$$

$$B(j) \to \beta_j\left(p_{j1}B(1) + p_{j2}B(2) + \cdots + p_{jm}B(m)\right)\rho + \gamma_j$$

To match the stacked system from equation (8), the ALM and associated T-map can also be stacked so that under the stacked PLM, $\hat{y_t} = A + Br_t$, $B = (B(1)', \ldots, B(m)')', and\ A = (A(1)', \ldots, A(m)')'$. The T-map is given by

$$T(A, B)' = \left(\left(\oplus_{j=1}^{m} \beta_j\right)(P \otimes I_n)A, \left(\oplus_{j=1}^{m} \beta_j\right)(P \otimes I_n)B\rho + \gamma\right),$$

and the RDE is a fixed point of the stacked T-map. Note, $T: \mathbb{R}^{nmx1} \oplus \mathbb{R}^{nmxk} \to \mathbb{R}^{nmx1} \oplus \mathbb{R}^{nmxk}$.

The eigenvalues of the Jacobian matrices

$$DT_A = \left(\oplus_{j=1}^{m} \beta_j\right)(P \otimes I_n)$$

$$DT_B = \rho' \otimes [\left(\oplus_{j=1}^{m} \beta_j\right)(P \otimes I_n)]$$

Provide the results for E-Stability, i.e. E-Stability requires real parts less than one, so that the E-stability condition is implied by the CLDC.

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ABSTRACT

ESSAYS IN ADAPTIVE LEARNING AND MEAN-SQUARE STABILITY IN REGIME SWITCHING MODELS

by

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Advisor: Dr. Tatsuma Wada

Major: Economics

Degree: Doctor of Philosophy

The first chapter of this dissertation analyzes the necessary and sufficient conditions for stability under recurring structural changes. Using a finite state Markov process to model stochastically evolving, state-dependent parameters I find that by employing the conditions unique to mean-square stability, the minimum state variable (MSV) solution, found in non-linear models of this reduced form, is also stable in the learning sense. However, the choice of parameter values limits the robustness of this result. Furthermore, to illustrate this outcome I develop empirical results for a model similar to Cagan's 1956 work on hyperinflation for Germany and the United States. I find that during the time of active currency market intervention, monetary policy was not mean-square stable for both the U.S. and Germany.

In the second chapter, I analyze if economic agents could have learned the policy decisions of the Plaza and Louvre accords. New techniques in Markov switching Adaptive Learning models (MSAL), shows that economic agents would not have learned the rational expectations outcomes of exchange rate interventions and therefore contributed to exchange rate overshooting and excess volatility during this time. These finding help to

explain why forecasts of short-term exchange rates have historically been poor while long-run forecasts do much better at matching the data.

The third chapter analyzes empirical data from the forward exchange rate premium to interpret the puzzle, made famous by Fama, using Markov Switching Adaptive Learning (MSAL) techniques. This chapter addresses the need for using Mean-Square Stability as the criterion for stability rather than traditional stability conditions. Moreover this chapter observes the possibility for a self-referential solution to occur under specific conditions similar to what is found empirically. Furthermore, this chapter is able to replicate the results typically found during the analysis using a Markov-switching constant gain model, indicating that economic agents may posses some form of bounded rationality or information asymmetry which produces the observed bias. A central tenant of this chapter is that agents facing a regime which tend to produce the forward premium bias present in most empirical applications even in the face of highly persistent fundamentals.

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